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Snapshots in Applied Complex Analysis

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Middle East Technical University
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R.P. Gilbert

Preface

The intension of this workshop was as well to instruct newcomers in the field of applied complex analysis about recent trends as to give them the opportunity to meet some of the experts. Among the younger participants were Algerians, Germans, Chinese, Turkish, Tajik, and Vietnamese. The experts came from Armenia, Belarus, China, Germany, France/Poland, Georgian, India, Italy, Russia, Tanzania, Turkey, USA. The workshop was financially supported by the Middle East Technical University in Ankara, the Scientific and Technical Research Council of Turkey and the Ankara Branch of the Turkish Mathematical Society. Their generous support is gratefully acknowledged. Without it the character of the workshop would have been completely different as most of the young generation and many experts in particular from the former SU and countries with weak economy would not have participated.

At the workshop some series of lectures and about 25 talks were given and the selection of manuscript in this volume reflects the range of topics but not the familiar and stimulating atmosphere at the meeting. The comfortable hosting at the guest house and dormitories of the METU, the kind care taking of the local organizers and their helpers and the homogeneous composition of the 30 participants had created a friendly background for an exciting workshop.

The topics vary from complex ordinary differential equations and regular holomorphic systems to complex partial differential equations, from boundary value problems of Riemann, Haseman and Riemann-Hilbert type to Hele Shaw flows and porosity of cancellous bones, from domain perturbation problems to optimization of fixed point methods, from quaternionic to Clifford analysis.

Growth estimations in a certain angular domain for entire finite order solutions to inhomogeneous linear complex ordinary differential equations with entire coefficients are provided under wider conditions as in the past. Motivated by investigations of the geometry of a-points of a meromorphic function the closedness of a- and b-points of an arbitrary polynomial for different a and b are measured and the sum of these measures for several pairs of different points estimated from above where the upper bound is independent of the number of pairs. Combining methods from singularity theory, complex analysis, and ordinary differential equations holonomic Picard-Fuchs systems of differential equations regular singular along hypersurfaces are studied. As an introduction to the theory of boundary value problems for complex partial differential equations of arbitrary order the basic problems of Schwarz, Dirichlet, and Neumann are solved for the inhomogeneous Cauchy-Riemann equation. The Schwarz problem is considered in Wiener domains for some particular quasilinear Bitsadze equation.

Eliptic systems of 2n second order equations with constant leading coefficients are treated in the upper half plane on the basis of the Bitsadze representation and the results applied to anisotropic plane elasticity. Three versions of the modified doubly-periodic second fundamental complete plane strain problem with relative displacements are formulated and solved via the Sherman transform the kernel function of which is given by the Weierstraß ζ-function.

More advanced boundary value problems in complex analysis are the Riemann-Hilbert and the Riemann problem, the first being a generalization of the Schwarz problem and in particular situations related to the latter. The Riemann problem is a special form as well of the general $\mathbb{R}$-linear conjugation problem as of the Haseman problem where also boundary shifts are involved. An $\mathbb{R}$-linear conjugation problem for elliptic functions in a disc is solved in form of Eisenstein series via a transformation to a functional equation. The result is applied to calculate the effective conductivity of a composite material with circular inclusions. A general formulation of the Riemann-Hilbert problem on almost complex manifolds is given within the theory of geometric and topological aspects of holomorphic curves in a loop spaces. Reduction to related problems for analytic functions is used to treat the Haseman boundary value problem for bianalytic functions.

The celebrated Hele-Shaw moving boundary value problem is considered in a doubly connected domain in decomposing it in a Schwarz boundary value problem and an abstract Cauchy-Kowalewski problem. Local existence and uniqueness of the classical solution is proved. The dependence of the porosity of cancellous bones on the measured ultrasonic wave is numerically computed. Here the Biot system of compressional wave equations for a dispersive dissipative fluid-saturated porous medium in the time domain is applied.

With functional analytic and potential theoretic methods domain perturbation problems for boundary value problems for homogeneous elliptic equations are investigated. In particular the Dirichlet and the Neumann problems for the Laplace equation are considered in the n-dimensional Euclidean space. As is well-known the solution of the Picard-Lindelöf integral equation is locally uniquely solvable. This situation often occurs when initial or boundary value problems are solved iteratively. Here the problem is studied how this "local" can be optimized.

Complex analytic methods are not restricted to plane problems. Quaternionic, octonionic, and Clifford analysis are theories efficiently used in mathematical physics where complex analytic methods are applied. Recently higher order Helmholtz operators and their factors were considered in quaternionic analysis as were higher order Laplace operators and their factors in Clifford analysis. On the basis of the quaternionic Stokes formula orthogonal decompositions of the Lebesgue space of square integrable quaternionic-valued functions are given with respect to the poly-α-hyperholomorphic and the polymetaharmonic functions. Higher order Cauchy-Pompeiu representations in Clifford analysis are known for powers of the Dirac operator D and the Laplace operator $\Delta = D\bar{D}$. Here such integral representations are developed for arbitrary operators $D^m\bar{D}^n$ for $m, n \in \mathbb{N}$. Also an addendum to those representation formulas in a universal Clifford algebra in case of an even-dimensional Eulidean space of dimension n is given with respect to D^k for $n \leq k$. Higher order Cauchy-Pompeiu representations for functions of $n, 1 \leq n$, complex variables are available in particular for polydomains. A general formula is given in connection with the operators $\partial_{z_\kappa}^{k_\kappa}\partial_{\bar{z}_\lambda}^{l_\lambda}, k_\kappa, l_\lambda \in \mathbb{N}, 1 \leq \kappa, \lambda \leq n$. Properties of the polyharmonic Green function for the upper half of the complex plane are studied.

The manuscripts were composed by Mr Kerem Kaskalouglu, from the Department of Mathematics of the METU and by Mrs. Barbara G. Wengel at the I. Mathematical Institute of FU Berlin. Thanks are due to both of them. The editors are also grateful to Professor A. Anastassiou for having offered to publish this volume in his new journal JAFA as first issues. They wish JAFA a properous future.

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Growth of solutions of complex non-homogeneous linear differential equations

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Abstract

In this paper, we obtain some precise estimates of the growth of finite order solutions of the differential equation $f^{(k)} + A_{k-1}f^{(k-1)} + ... + A_1 f^{'} + A_0 f = F$, where $A_0, ..., A_{k-1}$ and $F \not\equiv 0$ are entire functions, and there exists an A_s $(1 \leq s \leq k-1)$ that is dominant in the sense that it has larger growth than any other A_j $(0 \leq j \leq k-1,\ j \neq s)$.

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1 Introduction and statement of result

Let $\rho(f)$ denote the order of an entire function f, that is,

$$\rho(f) = \overline{\lim_{r\to+\infty}} \frac{\log T(r,\ f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log\log M(r,\ f)}{\log r}, \tag{1}$$

where $T(r,f)$ is the Nevanlinna characteristic function of f (see [5]), and $M(r,f) = \max_{|z|=r} |f(z)|$.
For $k \geq 2$ we consider the non-homogeneous linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \ldots + A_1(z) f' + A_0(z) f = F(z), \tag{2}$$

where $A_0(z), \ldots, A_{k-1}(z)$ and $F(z) \not\equiv 0$ are entire functions. It is well-known that all solutions of equation (2) are entire functions and if at least one coefficient $A_s(z)$ is transcendental, then most of the solutions are of infinite order. On the other hand, there exist equations of this form that possess one or more solutions of finite order. For example: $f(z) = e^z + 2$ satisfies $f''' + e^{-z} f'' - e^z f' + e^z f = 3e^z + 1$. It is also known that if there exists one A_s $(0 \leq s \leq n-1)$ such that A_s is transcendental with

$$\max\{\sigma(A_j)\ \ (j \neq s), \sigma(F)\} < \sigma(A_s) \leq 1/2,$$

then every transcendental solution f of (1.2) is of infinite order ([6]). For some general and related results of n-th order non-homogeneous linear differential equations with entire coefficients, we refer the reader to [4]. Our starting point for this paper is a result due to Gundersen in [2]

Theorem A ([2], p. 417) *Let $A(z)$ and $B(z) \not\equiv 0$ be entire functions such that for real constants α, β, θ_1 and θ_2 where $\alpha > 0$, $\beta > 0$ and $\theta_1 < \theta_2$, we have*

$$|A(z)| \geq \exp\left\{(1 + o(1))\, \alpha |z|^{\beta}\right\} \tag{3}$$

and

$$|B(z)| \leq \exp\left\{o(1)\, |z|^{\beta}\right\} \tag{4}$$

a $z \to \infty$ in $\theta_1 \leq \arg z \leq \theta_2$. Let $\varepsilon > 0$ be a given small constant, and let $S(\varepsilon)$ denote the angle $\theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon$.

If $f \not\equiv 0$ is a solution of finite order of the linear differential equation

$$f'' + A(z) f' + B(z) f = 0, \tag{5}$$

then the following conditions hold
(i) *There exists a constant $b \neq 0$ such that $f(z) \to b$ as $z \to \infty$ i $S(\varepsilon)$. Furthermore,*

$$|f(z) - b| \leq \exp\left\{-(1 + o(1))\alpha |z|^{\beta}\right\} \tag{6}$$

as $z \to \infty$ in $S(\varepsilon)$.
(ii) *For each integer $k \geq 1$*

$$\left|f^{(k)}(z)\right| \leq \exp\left\{-(1 + o(1))\alpha |z|^{\beta}\right\} \tag{7}$$

as $z \to \infty$ in $S(\varepsilon)$.

This result has been recently generalized to higher order by the author and Hamani as follows (see [1]) :

Theorem B ([1]). *Let $A_0(z), \ldots, A_{k-1}(z)$ with $A_0(z) \not\equiv 0$ be entire functions such that for real constants α, β, θ_1 and θ_2 where $\alpha > 0$, $\beta > 0$ and $\theta_1 < \theta_2$, we have*

$$|A_1(z)| \geq \exp\left\{(1 + o(1))\alpha |z|^{\beta}\right\} \tag{8}$$

and

$$|A_j(z)| \leq \exp\left\{o(1)|z|^{\beta}\right\} \quad (j = 0, 2, ..., k-1) \tag{9}$$

as $z \to \infty$ in $\theta_1 \leq \arg z \leq \theta_2$. Let $\varepsilon > 0$ be a given small constant, and let $S(\varepsilon)$ denote the angle $\theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon$. If $f \not\equiv 0$ is a solution of finite order of the linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = 0, \tag{10}$$

then the following conditions hold
(i) *There exists a constant $b \neq 0$ such that $f(z) \to b$ as $z \to \infty$ in $S(\varepsilon)$. Furthermore,*

$$|f(z) - b| \leq \exp\left\{-(1 + o(1))\alpha |z|^{\beta}\right\} \tag{11}$$

as $z \to \infty$ in $S(\varepsilon)$.
(ii) *For each integer $m \geq 1$*

$$\left|f^{(m)}(z)\right| \leq \exp\left\{-(1 + o(1))\alpha |z|^{\beta}\right\} \tag{12}$$

as $z \to \infty$ *in* $S(\varepsilon)$.

The main purpose of this paper is to extend results of Theorem B to the non-homogeneous linear differential equation (2) in the following theorem, in which the dominating coefficient $A_1(z)$ is replaced by $A_s(z)$. The method used in this paper will be quite different from that in the proof of Theorem B (see [1]).

Theorem 1 *Suppose that* $A_0(z), \ldots, A_{k-1}(z)$ *and* $F(z) \not\equiv 0$ *are entire functions such that for real constants* α, β, θ_1 *and* θ_2 *where* $\alpha > 0$, $\beta > 0$ *and* $\theta_1 < \theta_2$, *we have for some* $s = 1, ..., k-1$,

$$|A_s(z)| \geq \exp\left\{(1 + o(1))\,\alpha |z|^{\beta}\right\} \tag{13}$$

and for all $j = 0, \ldots, s-1, s+1, ..., k-1$

$$\max\{|A_j(z)|\ , |F(z)|\} \leq \exp\left\{o(1)\,|z|^{\beta}\right\} \tag{14}$$

as $z \to \infty$ *in* $\theta_1 \leq \arg z \leq \theta_2$. *For given* $\varepsilon > 0$ *small enough, let* $S(\varepsilon)$ *denote the angle* $\theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon$. *If* f *is a transcendental solution of equation* (2) *with* $\rho(f) < +\infty$, *then the following conditions hold:*

(i) *There exists a constant* b_{s-1} *such that* $f^{(s-1)}(z) \to b_{s-1}$ *as* $z \to \infty$ *in* $S(\varepsilon)$. *Indeed,*

$$\left|f^{(s-1)}(z) - b_{s-1}\right| \leq \exp\left\{-(1 + o(1))\,\alpha |z|^{\beta}\right\} \tag{15}$$

as $z \to \infty$ *in* $S(\varepsilon)$.

(ii) *For each integer* $m \geq s$

$$\left|f^{(m)}(z)\right| \leq \exp\left\{-(1 + o(1))\,\alpha |z|^{\beta}\right\} \tag{16}$$

as $z \to \infty$ *in* $S(\varepsilon)$.

2 Preliminary Lemmas

We need the following lemmas for the proof of our theorem.

Lemma 1 ([3], p. 89) *Let* f *be a transcendental entire function of finite order* ρ, *let* $\Gamma = \{(k_1, j_1), (k_2, j_2), \ldots, (k_m, j_m)\}$ *denote a finite set of distinct pairs of integers that satisfy* $k_i > j_i \geq 0$ $(i = 1, \ldots, m)$, *and let* $\varepsilon > 0$ *be a given constant. Then there exists a set* $E \subset [0, 2\pi)$ *that has linear measure zero, such that if* $\psi_0 \in [0, 2\pi) - E$, *then there is*

a constant $R_0 = R_0(\psi_0) > 1$ *such that for all* z *satisfying* $\arg z = \psi_0$ *and* $|z| \geq R_0$, *and for all* $(k, j) \in \Gamma$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{17}$$

Lemma 2 ([7]) *Let* $f(z)$ *be an entire function and suppose that* $\left|f^{(k)}(z)\right|$ *is unbounded on some ray* $\arg z = \theta$. *Then there exists an infinite sequence of points* $z_n = r_n e^{i\theta}$ $(n = 1, 2, \ldots)$, *where* $r_n \to +\infty$, *such that* $f^{(k)}(z_n) \to \infty$ *and*

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right| \leq \frac{1}{(k-j)!}(1 + o(1))|z_n|^{k-j} \quad (j = 0, \ldots, k-1). \tag{18}$$

3 Proof of Theorem 1

Suppose that f is a solution of (2) with $\rho(f) < +\infty$. Set $\rho = \rho(f)$. Then by Lemma 1, there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E$, then for all $k > s \geq 1$, and all $j = s+1, \ldots, k$,

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \leq |z|^{(j-s)(\rho-1+\varepsilon)} \leq |z|^{(k-s)\rho} \quad (0 < \varepsilon < 1) \tag{19}$$

as $z \to \infty$ along $\arg z = \psi_0$.
Suppose now, for a while, that $\left|f^{(s)}(z)\right|$ is unbounded on some ray $\arg z = \phi_0$ where $\phi_0 \in [\theta_1, \theta_2] - E$. Then by Lemma 2, there exists an infinite sequence of points $z_n = r_n e^{i\phi_0}$, where $r_n \to +\infty$ such that $f^{(s)}(z_n) \to \infty$ and

$$\left|\frac{f^{(j)}(z_n)}{f^{(s)}(z_n)}\right| \leq \frac{1}{(s-j)!}(1 + o(1))|z_n|^{s-j} \leq 2|z_n|^s \quad (j = 0, \ldots, s-1) \tag{20}$$

as $z_n \to \infty$. From (2), we next conclude that

$$|A_s(z)| \leq \left|\frac{f^{(k)}}{f^{(s)}}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f^{(s)}}\right| + \ldots + |A_{s+1}(z)|\left|\frac{f^{(s+1)}}{f^{(s)}}\right|$$

$$|A_{s-1}(z)|\left|\frac{f^{(s-1)}}{f^{(s)}}\right| + \ldots + |A_1(z)|\left|\frac{f'}{f^{(s)}}\right| + |A_0(z)|\left|\frac{f}{f^{(s)}}\right| + \left|\frac{F}{f^{(s)}}\right|. \tag{21}$$

Using (19)-(21), (13) and (14), we obtain that

$$\exp\left\{\left(1+o\left(1\right)\right)\alpha\left|z_n\right|^{\beta}\right\}\leq 2\left|z_n\right|^{s}s\exp\left\{o\left(1\right)\left|z_n\right|^{\beta}\right\}$$

$$+\left|z_n\right|^{(k-s)\rho}+(k-s-1)\left|z_n\right|^{(k-s)\rho}\exp\left\{o\left(1\right)\left|z_n\right|^{\beta}\right\}+\frac{\exp\left\{o\left(1\right)\left|z_n\right|^{\beta}\right\}}{\left|f^{(s)}\left(z_n\right)\right|}. \tag{22}$$

It follows from (22) that

$$\frac{\exp\left\{\left(1+o\left(1\right)\right)\alpha\left|z_n\right|^{\beta}\right\}}{\left|z_n\right|^{s}\left|z_n\right|^{(k-s)\rho}\exp\left\{o\left(1\right)\left|z_n\right|^{\beta}\right\}}\leq\frac{2s}{\left|z_n\right|^{(k-s)\rho}}+\frac{1}{\left|z_n\right|^{s}\exp\left\{o\left(1\right)\left|z_n\right|^{\beta}\right\}}$$

$$+\frac{(k-s-1)}{\left|z_n\right|^{s}}+\frac{1}{\left|z_n\right|^{s}\left|z_n\right|^{(k-s)\rho}\left|f^{(s)}\left(z_n\right)\right|}.$$

Since $f^{(s)}\left(z_n\right)\to\infty$ as $z_n\to\infty$, we will obtain a contradiction. Therefore, $\left|f^{(s)}\left(z\right)\right|$ is remains bounded on any ray $\arg z=\phi$ where $\phi\in[\theta_1,\theta_2]-E$. It then follows by a standard application of the classical Phragmén-Lindelöf theorem [8, p. 214], that there exists a constant $M>0$ such that

$$\left|f^{(s)}\left(z\right)\right|\leq M \tag{23}$$

for all $z\in S\left(\varepsilon\right)$.

If $\theta_0\in[\theta_1+\varepsilon,\theta_2-\varepsilon]-E$, then when $\arg z=\theta_0$, we obtain for all $m<s$, by $(s-m)$-fold iterated integration along the ray under consideration,

$$f^{(m)}\left(z\right)=f^{(m)}\left(0\right)+f^{(m+1)}\left(0\right)z+\ldots+\frac{1}{(s-m-1)!}f^{(s-1)}\left(0\right)z^{s-m-1}$$

$$+\int_0^z\ldots\int_0^{\zeta}\int_0^{\xi}f^{(s)}\left(t\right)dtd\xi\ldots du. \tag{24}$$

Therefore, by an elementary triangle inequality and (23), we obtain from (24)

$$\left|f^{(m)}\left(z\right)\right|\leq\left|f^{(m)}\left(0\right)\right|$$

$$+\left|f^{(m+1)}\left(0\right)\right|\left|z\right|+\ldots+\frac{1}{(s-m-1)!}\left|f^{(s-1)}\left(0\right)\right|\left|z\right|^{s-m-1}$$

$$+M\int_0^z\ldots\int_0^{\zeta}\int_0^{\xi}\left|dt\right|\left|d\xi\right|\ldots\left|du\right|=O\left(\left|z\right|^{s-m}\right). \tag{25}$$

Writing now (2) in the form

$$|A_s(z)|\left|f^{(s)}\right| \leq |F| +$$

$$\left(\left|\frac{f^{(k)}}{f^{(s)}}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f^{(s)}}\right| + \ldots + |A_{s+1}(z)|\left|\frac{f^{(s+1)}}{f^{(s)}}\right|\right)\left|f^{(s)}\right|$$

$$+ |A_{s-1}(z)|\left|f^{(s-1)}\right| + \ldots + |A_1(z)|\left|f'\right| + |A_0(z)||f|. \tag{26}$$

By using (19), (23), (25), (13) and (14), we obtain from (26)

$$\exp\left\{(1+o(1))\,\alpha|z|^{\beta}\right\}\left|f^{(s)}\right| \leq \exp\left\{o(1)|z|^{\beta}\right\}$$

$$+ |z|^{(k-s)\rho}\left(1+(k-s-1)\exp\{o(1)|z|^{\beta}\}\right)\left|f^{(s)}\right|$$

$$+ (O(|z|^{s}) + \ldots + O(|z|))\exp\{o(1)|z|^{\beta}\}$$

$$\leq \exp\left\{o(1)|z|^{\beta}\right\} + |z|^{(k-s)\rho}\left(1+(k-s-1)\exp\left\{o(1)|z|^{\beta}\right\}\right)M$$

$$+ (O(|z|^{s}) + \ldots + O(|z|))\exp\left\{o(1)|z|^{\beta}\right\} \tag{27}$$

as $z \to \infty$ along $\arg z = \theta_0$. By (3.9), we conclude that

$$\left|f^{(s)}(z)\right|$$

$$\leq \frac{\exp\left\{o(1)|z|^{\beta}\right\} + |z|^{(k-s)\rho}\left(1+(k-s-1)\exp\left\{o(1)|z|^{\beta}\right\}\right)M}{\exp\left\{(1+o(1))\,\alpha|z|^{\beta}\right\}}$$

$$+\frac{(O(|z|^{s}) + \ldots + O(|z|))\exp\left\{o(1)|z|^{\beta}\right\}}{\exp\left\{(1+o(1))\,\alpha|z|^{\beta}\right\}} \leq \exp\left\{-(1+o(1))\,\alpha|z|^{\beta}\right\}. \tag{28}$$

as $z \to \infty$ along $\arg z = \theta_0$. By using an application of the Phragmén-Lindelöf theorem to (28), it can be deduced that

$$\left|f^{(s)}(z)\right| \leq \exp\left\{-(1+o(1))\,\alpha|z|^{\beta}\right\} \tag{29}$$

as $z \to \infty$ in $S(2\varepsilon)$. This proves the second assertion for $m = s$.

Now let $z \in S(3\varepsilon)$ where $|z| > 1$, let γ be a circle of radius $r = 1$ with center at z, and let $m > s$ be an integer. Then by the Cauchy integral formula and (29), we obtain as $z \to \infty$ in $S(3\varepsilon)$,

$$\left|f^{(m)}(z)\right| \leq \frac{(m-s)!}{2\pi} \oint_{\gamma} \frac{\left|f^{(s)}(u)\right|}{|u-z|^{m-s+1}} |du|$$

$$\leq \frac{(m-s)!}{2\pi}.2\pi \exp\left\{-(1+o(1))\,\alpha\,(|z|-1)^{\beta}\right\}$$

$$\leq \exp\left\{-(1+o(1))\,\alpha\,|z|^{\beta}\right\}. \tag{30}$$

This proves the second assertion for $m > s$.
Now fix θ where $\theta_1 + \varepsilon \leq \theta \leq \theta_2 - \varepsilon$, and set

$$a_{s-1} = \int_0^{+\infty} f^{(s)}\left(te^{i\theta}\right) e^{i\theta} dt, \tag{31}$$

where we note that we obtain from (29) the existence of a_{s-1} and that $a_{s-1} \in \mathbf{C}$. Indeed, integrating $f^{(s)}(u)$ along the sector boundary $0 \to R\ e^{i\psi} \to R\ e^{i\theta} \to 0$, by using (29) and Cauchy's theorem to conclude that the integral of $f^{(s)}(u)$ over the arc $\left[Re^{i\psi}, Re^{i\theta}\right]$ tends to zero as $R \to +\infty$, the independence from θ immediately follows. Let $z = |z|\, e^{i\,\psi}$ where $\theta_1 + \varepsilon \leq \psi \leq \theta_2 - \varepsilon$. Then by (31), we obtain

$$f^{(s-1)}(z) - f^{(s-1)}(0) - a_{s-1} = \int_0^{z} f^{(s)}(u)\, du - \int_0^{+\infty} f^{(s)}\left(te^{i\psi}\right) e^{i\psi} dt$$

$$= \int_0^{z} f^{(s)}(u)\, du - \left(\int_0^{|z|} f^{(s)}\left(te^{i\psi}\right) e^{i\psi} dt + \int_{|z|}^{+\infty} f^{(s)}\left(te^{i\psi}\right) e^{i\psi} dt \right)$$

$$= -\int_{|z|}^{+\infty} f^{(s)}\left(te^{i\psi}\right) e^{i\psi} dt \tag{32}$$

Using (29) and (32), we conclude that

$$\left|f^{(s-1)}(z) - f^{(s-1)}(0) - a_{s-1}\right| = \left| \int_{|z|}^{+\infty} f^{(s)}\left(te^{i\psi}\right) e^{i\psi} dt \right|$$

$$\leq \int_{|z|}^{+\infty} \exp\left\{-(1+o(1))\,\alpha t^{\beta}\right\} dt$$

$$\leq \frac{1}{(1+o(1))\,\alpha\beta\frac{|z|^{\beta-1}}{2}\exp\left\{(1+o(1))\,\alpha\frac{|z|^{\beta}}{2}\right\}} \int_{|z|}^{+\infty} \frac{(1+o(1))\,\alpha\beta\frac{t^{\beta-1}}{2}}{\exp\left\{(1+o(1))\,\alpha\frac{t^{\beta}}{2}\right\}} dt$$

$$\leq \frac{1}{(1+o(1))\,\alpha\beta\frac{|z|^{\beta-1}}{2}\exp\left\{(1+o(1))\,\alpha\frac{|z|^{\beta}}{2}\right\}} \exp\left\{-(1+o(1))\,\alpha\frac{|z|^{\beta}}{2}\right\}$$

$$\leq \exp\left\{-(1+o(1))\,\alpha\,|z|^{\beta}\right\} \tag{33}$$

as $z \to \infty$ in $S(\varepsilon)$, where $b_{s-1} = f^{(s-1)}(0) + a_{s-1}$. We note also that $f^{(s-1)}(z) \to b_{s-1}$ as $z \to \infty$ in $S(\varepsilon)$ from (33). The proof of Theorem 1 is complete.

Next, we give two examples that illustrate Theorem 1.

Example 1 Consider the differential equation

$$f''' + ze^{-z}f'' + e^{z}f' - (e^{z}-1)\,f = (z+1)\,e^{z}. \tag{34}$$

In this equation, for $z = re^{i\theta}$ $(r \to +\infty)$ and $\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{6}$ we have

$$|A_2(z)| = |ze^{-z}| = r\exp(-r\cos\theta) \geq \exp\left((1+o(1))\,\frac{\sqrt{2}}{2}r\right)$$

$$|A_1(z)| = |e^{z}| \leq \exp(r\cos\theta) \leq \exp(o(1)\,r)$$

$$|A_0(z)| = |-(e^{z}-1)| \leq 1 + \exp(r\cos\theta) \leq \exp(o(1)\,r)$$

$$|F(z)| = |(z+1)\,e^{z}| = (r+1)\exp(r\cos\theta) \leq \exp(o(1)\,r).$$

Hence the conditions (13) and (14) of Theorem 1 are verified ($\alpha = \frac{\sqrt{2}}{2}$, $\beta = 1$), with $A_2(z) = ze^{-z}$ is the dominating coefficient. The function $f(z) = e^{z} - z$ with $\rho(f) = 1$ satisfies equation (34) and the relations (15), (16) with $b_1 = -1$.

Example 2 Consider the differential equation

$$f''' + e^{z}f'' + e^{-z}f' - e^{z}f = 1 + e^{z}. \tag{35}$$

In this equation, for $z = re^{i\theta}$ $(r \to +\infty)$ and $\frac{2\pi}{3} \leq \theta \leq \frac{3\pi}{4}$ we have

$$|A_1(z)| = |e^{-z}| = \exp(-r\cos\theta) \geq \exp\left((1+o(1))\frac{r}{2}\right)$$
$$|A_0(z)| = |-e^{z}| = \exp(r\cos\theta) \leq \exp(o(1)\,r)$$
$$|A_2(z)| = |e^{z}| = \exp(r\cos\theta) \leq \exp(o(1)\,r)$$
$$|F(z)| = |1+e^{z}| \leq 1+\exp(r\cos\theta) \leq \exp(o(1)\,r).$$

It is easy to see that the conditions (13) and (14) of Theorem 1 are verified ($\alpha = \frac{1}{2}, \beta = 1$), with $A_1(z) = e^{-z}$ is the dominating coefficient. The function $f(z) = e^{z}$ with $\rho(f) = 1$ satisfies equation (35) and the relations (15), (16) with $b_0 = 0$.

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On closeness of a- and b-points of arbitrary polynomials

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Abstract

In this note a new type of generality in the theory of polynomials is established which gives upper bounds for the distances between a-points and b-points of arbitrary complex polynomials.

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1 Statements

First we define a *total distance m between two sets of given complex numbers $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$.* Let X^* (Y^*) be the totality of all possible permutations of the first (second) set. Composing for two elements

$$\{x_1, x_2, ..., x_n\} \in X^* \text{ and } \{y_1, y_2, ..., y_n\} \in Y^*$$

the following sum $\sum_{\nu=1}^{n} |x_i - y_i|$ we consider those element(s) of X^* and Y^* for which this sum is minimal. This minimum we denote by m. A natural characteristic of the mutual locations of these sets is also $\frac{m}{n}$ which we call *averaged distance between the above sets.* The introduced magnitudes have clear geometric meanings.

Due to the main theorem of algebra for an arbitrary complex polynomial $P(z) := c_0 z^n + c_1 z^{n-1} + c_2 z^{n-2} + ... + c_n$ and arbitrary complex values a and b the cardinalities of a-points and b-point are equal to n and, therefore, we can consider the magnitude m in this case (denote $m := m(a, b, P)$).

Theorem 1 *For an arbitrary complex polynomial $P(z)$ and for arbitrary complex values $a_1, b_1, a_2, b_2, ..., a_q, b_q$ lying on the segment $[a_1, b_q]$ in the same order the estimation*

$$\sum_{\nu=1}^{q} m(a_\nu, b_\nu, P) < 24 \left| \frac{a_1 - b_q}{c_0} \right|^{1/n} n \tag{1}$$

is valid.

From (1) immediately follows

Corollary 1 *If the polynomial $P(z)$ is monic (that is $P^*(z) := z^n + c_1 z^{n-1} + c_2 z^{n-2} + ... + c_n$) then $m(a, b, P) < 24 |a - b|^{1/n} n$.*

Therefore $m(0, 1, P) < 24n$. Thus, the averaged distance $m(a, b, P)/n$ between the zeros and 1-points is less than 24.

Remark 1 (sharpness) For $P(z) = z^n$, $a = 0$ and $b = 1$ we have $m(0, 1, z^n) = n$ and $m(0, 1, z^n)/n = 1$ so that the inequality (1) is sharp up to this constant 24. On the other hand if we take $P(z) = c_0 z^n$, $a = 0$ then for arbitrary value b (with large $|b|$) we have $m(0, b, c_0 z^n) = |b/c_0|^{1/n} n$ meantime due to (1) we have $m(0, b, c_0 z^n) < 24 |b/c_0|^{1/n} n$. This shows that inequality (1) is asymptotically sharp (up to this constant 24) when $|b| \to \infty$.

Remark 2 (connections with the proximity property) For meromorphic functions in the complex plane the so called proximity prop-

erty of a-points has been established [1] studying the geometry (mutual locations) of a- and b-points; meantime the classical Nevanlinna and Ahlfors theories study the quantities of a- and b-points only. A similar progress has lead to reexamining many problems in different fields of mathematics [2], where instead of earlier quantitative studies of a-points we particularly offer to study problems considering the geometry of a-points. One of the problems in [2] (problem 32) is to establish theproximity property for polynomials. In fact Theorem 1 is closely connected with this problem. It is interesting that the proximity property has been established first for more complicated meromorphic functions and only later on we consider the phenomenon for the comparatively simple class of polynomials.

Below in fact we establish a more general possible version of this proximity property ([2], problem 32) for polynomials.

Let Γ be a curve passing trough different complex points $a_1, b_1, a_2, b_2, ..., a_q, b_q$ in the same order. Suppose that these points lie in a disk G with radius M and that the curve Γ belongs to G and is a smooth Jordan curve with finite absolute integral curvature $v(\Gamma)$: we mean that $v(\Gamma) := \int_\Gamma |k(s)|ds < \infty$, where $k(s)$ is the curvature of Γ at the point s.

Theorem 2 *For an arbitrary complex polynomial $P(z)$ the estimation*

$$\sum_{\nu=1}^{q} m(a_\nu, b_\nu, P) < 795(v(\Gamma)+1)\left(\frac{M}{|c_0|}\right)^{1/n} n \qquad (2)$$

is valid.

Remark 3 The right-hand side of (2) does not depend on the quantity q. This is similar to the second main theorem in Nevanlinna theory, where this phenomenon leads to the deficiency relation. In our case (2) leads to the simple conclusion that for arbitrary large totality of pairs (a_ν, b_ν) lying in G the above sum can be large only when the geometry of the points $a_1, b_1, a_2, b_2, ..., a_q, b_q$ is complicated; in other words any curve passing trough these points should have rather large absolute integral curvature.

2 Proofs

The proofs use a technique from [3]. According to inequalities (1.2.7) and (1.2.9) in [1], (see p. 20-21) , for any meromorphic functions $w(z)$

in the closure of a domain D with piecewise smooth boundary ∂D and any straight line Γ

$$L(D,\Gamma,w) \leq \frac{2\sqrt{2}}{\pi}\int\int_D \left|\frac{w''}{w'}\right| d\sigma + \sqrt{2}l(D) \tag{3}$$

is valid, where $L(D,\Gamma,w)$ is the length of Γ-lines of w in D, $l(D)$is the length of ∂D. Recall that Γ-lines are the preimages $w^{-1}(\Gamma)$.

It is known that for a monic polynomial P of degree n the set $\{z:\ |P(z)|\leq M\}$ is contained in the union of some disks d_k, $k=1,2,...,k^*\leq n$, the sum of whose radii r_k is $2eM^{1/n}$. This result is due to H. Cartan, see [4], p. 19; Ch. Pommerenke [5] improved the constant $2e$ to 2.59. With this M the union $\cup_k\left\{\overline{d_k}\right\}$, ($\overline{x}$ means the closure), consists, clearly, of some closed non overlapping domains D_j, $j=1,2,...k^*\leq n$, with total length of the boundary $\leq 2\pi\sum_k r_k \leq 5.18\pi\sqrt{2}M^{1/n}$. Denoting $D^*:=\cup_j D_j$ and applying (3) to P in these domains D_j we have

$$\begin{aligned}
&L(D^*,\Gamma,P)=\sum_{j=1}^{k^*}L(D_j,\Gamma,P)\\
&=\sum_{j=1}^{k^*}\left\{\frac{2\sqrt{2}}{\pi}\int\int_{D_j}\sum_{t=1}^{n-1}\left|\frac{1}{z-z_t}\right|d\sigma+\sqrt{2}l(D_j)\right\}\\
&\leq\frac{2\sqrt{2}}{\pi}\int\int_{D^*}\sum_{t=1}^{n-1}\left|\frac{1}{z-z_t}\right|d\sigma+\sqrt{2}\sum_{j=1}^{k^*}l(D_j)\\
&\leq\frac{2\sqrt{2}}{\pi}\int\int_{D^*}\sum_{t=1}^{n-1}\left|\frac{1}{z-z_t}\right|d\sigma+5.18\pi\sqrt{2}M^{1/n},
\end{aligned}\tag{4}$$

where z_t are zeros of the derivative P'. It is easy to see that if $|D^*|$ is the area of D^* then denoting $z-z_t=\rho e^{i\varphi}$ we have

$$\int\int_{D^*}\left|\frac{1}{z-z_t}\right|d\sigma=\int\int_{D^*}d\rho d\varphi\leq\int\int_{D_t^*}d\rho d\varphi, \tag{5}$$

where D_t^* is the disk with center in z_t and with $|D_t^*|=|D^*|$. Indeed, for any two domains

$$d:=\{z:\varphi_1<\arg z<\varphi_2,\ \rho_1<|z-z_t|<\rho_2\}$$

and

$$d':=\{z:\varphi_1'<\arg z<\varphi_2',\ \rho_1'<|z-z_t|<\rho_2'\}$$

with equal areas and $|\rho_2 - \rho_1| = |\rho_2' - \rho_1'|$ we have

$$\int\int_d d\rho d\varphi \leq \int\int_{d'} d\rho d\varphi$$

as soon as $\rho_1' < \rho_1$. Therefore dividing D^* into similar small domains d (we can take them as small as we please) and transferring them into domains d' with $\rho_1' < \rho_1$ we can compose from these disks d the disk D_t^* satisfying $|D_t^*| = |D^*|$ for which, due to the above inequality, (5) is valid.

Since the radius of D_t^* is less than or equal to $\sum d_k \leq 2.59M^{1/n}$ we conclude that $\int\int_{D_t^*} d\rho d\varphi \leq 5.18\pi M^{1/n}$ so that from (4) and (5) follows

$$L(D^*, \Gamma, P) \leq \frac{2\sqrt{2}}{\pi} 5.18\pi M^{1/n}(n-1) + 5.18\pi\sqrt{2}M^{1/n} <$$

$$5.18\pi\sqrt{2}M^{1/n}n < 24M^{1/n}n. \tag{6}$$

Suppose now that the values a and b lie on the straight line Γ. Let Γ^* be the segment on Γ with endpoints a and b . Consider the functions $P^* := P - \frac{a+b}{2}$ and note that a-points and b-points of P coincide respectively with $\frac{a-b}{2}$-points and $\frac{b-a}{2}$-points of P^*. Consequently

$$m(a, b, P) = m\left(\frac{a-b}{2}, \frac{b-a}{2}, P^*\right).$$

Further, Γ^*-lines of P^* can be considered as a totality of n curves γ_i, $i = 1, 2, ..., n$, each connecting one $\frac{a-b}{2}$-point of P^* with exactly one $\frac{b-a}{2}$-point P^* (with counting multiplicities). Therefore the total distance between these $\frac{a-b}{2}$-points and $\frac{b-a}{2}$-points is less than or equal to the length of the corresponding γ_i and consequently we obtain

$$m\left(\frac{a-b}{2}, \frac{b-a}{2}, P^*\right) \leq L(\mathbb{C}, \Gamma^*, P^*).$$

By taking now $a = a_\nu$ and $b = b_\nu$ and repeating the above reasoning for a_ν, b_ν and the corresponding curves γ_i we obtain

$$\sum_{\nu=1}^{q} m(a_\nu, b_\nu, P) \leq L(\mathbb{C}, \Gamma^*, P^*).$$

However Γ^* belongs to the set $\left\{P^* : |P^*| < \left|\frac{a_1 - b_q}{2}\right|\right\}$ and therefore $L(\mathbb{C}, \Gamma^*, P^*) = L(D^*, \Gamma^*, P^*)$, where D^* is defined as above with $\Gamma =$

Γ^*, $P = P^*$ and $M = \left|\frac{a_1 - b_q}{2}\right|$. Therefore by applying (6) we obtain the following inequality

$$L(D^*, \Gamma^*, P^*) < 24 \left|\frac{a_1 - b_q}{2}\right|^{1/n} n.$$

Thus we have

$$\sum_{\nu=1}^{q} m(a_\nu, b_\nu, P) \leq 24 \left|a_1 - b_q\right|^{1/n} n$$

for any monic polynomial P; immediately follows inequality (1) of Theorem 1 for arbitrary polynomials.

To prove Theorem 2 we need the following

Lemma 1 For any curve Γ lying in the disk G with radius M and center A and any monic polynomial P of degree n

$$L(\mathbb{C}, \Gamma, P - A) \leq 5.18\pi K_1(\Gamma) M^{1/n} n, \tag{7}$$

where $K_1(\Gamma) = 15.54\pi(v(\Gamma) + 1)$.

In fact this inequality has been proved in [3] when $M = 1$. To prove (7) for arbitrary M we need the following inequality (see [1], p.20) which is just a more general form of (3): for any meromorphic functions $w(z)$ in the closure of a domain D with piecewise smooth boundary ∂D and any curve Γ

$$L(D, \Gamma, w) \leq K_1(\Gamma) \int\int_D \left|\frac{w''}{w'}\right| d\sigma + K_1(\Gamma) l(D) \tag{8}$$

is valid. Note that

$$L(\mathbb{C}, \Gamma, P - A) = L(\tilde{G}, \Gamma, P - A) \leq L(D^*, \Gamma, P - A),$$

where $\tilde{G}$ is the preimage of G under the mapping by $P - A$ and D^* is defined as above for $P - A$. Applying now (8) with $D = D^*$, $w = P - A$ and taking into account that due to above inequalities $\int\int_{D^*} \left|\frac{P''}{P'}\right| d\sigma \leq 5.18\pi M^{1/n}(n-1)$ is valid and $l(D^*) \leq 5.18\pi M^{1/n}$ we obtain (7).

Further for Γ defined in Theorem 2 we have

$$\sum_{\nu=1}^{q} m(a_\nu, b_\nu, P) \leq L(\mathbb{C}, \Gamma, P) = L(\mathbb{C}, \tilde{\Gamma}, P - A),$$

where $\tilde{\Gamma} = \Gamma \backslash A$ (translation of Γ by A). For this curve $\tilde{\Gamma}$ and the function P Lemma 1 is valid so that we have

$$L(\mathbb{C}, \tilde{\Gamma}, P - A) \leq 5.18\pi K_1(\Gamma) M^{1/n} n < 795(v(\Gamma) + 1) M^{1/n} n.$$

Thus we obtain Theorem 2 for any monic polynomial P; inequality (2) of Theorem 2 immediately follows for arbitrary polynomials.

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Regular singular holonomic systems of differential equations with given integrals

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Abstract

We develop an approach to the study of holonomic systems of differential equations with regular singularities along hypersurfaces. In particular, we investigate Picard-Fuchs systems satisfied by the elliptic, hyperelliptic and versal integrals, describe the corresponding minimal equations and analyze properties of their solutions.
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1 Introduction

Simple methods for reduction of systems of differential equations to ordinary differential equations or vise versa are well-known; they are very often used and applied in the theory. In general, the elimination method is not very useful in the investigations of systems with unknown solutions since as a rule the equivalent equation is not simpler then the input system. However, when there is known some representation of solutions of the system, then the following problems appear naturally: what are the equations for a given solution, how are equations for different solutions related, what are the type (for instance, known or unknown) of these equations, etc.

This article is devoted to the study of some aspects of related problems in the context of the theory of holonomic systems of differential equations with regular singularities along hypersurfaces. In fact our approach is a combination of more or less known ideas from singularity theory, complex analysis and differential equations. At first we briefly discuss the elimination method for such systems, describe useful properties of the resulting equations and their solutions, consider some interconnections with the theory of holonomic D-modules, etc. Then we describe a natural interpretation of the notion of Gauss-Manin or Picard-Fuchs systems in the framework of deformation theory of singularities. One of the most interesting and important topic closely related to this subject is the theory of versal integrals developed by V.P. Palamodov [13] and his pupil Kh.M. Malikov [8]. To be more precise, they investigated Gauss-Manin or, equivalently, Picard-Fuchs systems associated with deformations of isolated hypersurface singularities. Here we study in detail the simplest case of A_μ-singularities. At first we recall some basic facts concerning systems which have the classical elliptic integrals as solutions. Then we analyze the systems satisfied by the versal integrals for A_μ-singularities and compute the corresponding minimal equations in some special cases. Further, we express the fundamental solution of the Gauss-Manin system for 2-parameter deformations of A_μ-singularities via classical hypergeometric functions depending on one variable. This is, in fact, a slightly corrected result obtained earlier by Kh. Malikov in [9]. In addition by direct computations we verify another result of his (loc. cite) which states that in case of the versal 3-parameter deformation of an A_3-singularity all components of the fundamental solution of the Gauss-Manin system are expressed via two-dimensional Horn's hyper-

geometric functions.

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2 Systems and equations

It is well-known that a nonsingular ordinary differential equation can be easy reduced to the normal system, that is, to a system containing equations of the following type only: $y_k' = f_k(x, \mathbf{y})$. For this it is necessary to redefine derivatives as new variables $y_k = y^{(k)}$. And conversely, an arbitrary system given in normal form can be reduced to one equation of the same order. However the following examples show that occasionally one can not consider unknown variables of the system as unknowns of the desirable equation. For example, let us consider the following system

$$\begin{cases} y' = x^2 z, \\ z' = x^2 y. \end{cases}$$

Eliminating z, one gets a singular equation for y

$$xy'' - 2y' - x^5 y = 0. \tag{1}$$

Otherwise, one can differentiate the first equation three times and eliminate z so that the singularity doesn't occur at all:

$$y^{(4)} - x^4 y'' - 10x^3 y' - 20x^2 y = 0. \tag{2}$$

It is evident that the operator of the latter equation is divided by the operator of the equation (1) :

$$\partial^4 - x^4\partial^2 - 10x^3\partial - 20x^2 = x^{-1}\partial^2(x\partial^2 - 2\partial - x^5), \tag{3}$$

where $\partial = \partial/\partial x$. In particular, it means that the ring of differential operators is non-factorial. It is not difficult to prove that the unknown

y doesn't satisfy a nonsingular equation of the third order. The solutions of the system are the following

$$\begin{cases} y = c_1 \cosh(x^3/3) + c_2 \sinh(x^3/3), \\ z = c_1 \sinh(x^3/3) + c_2 \cosh(x^3/3). \end{cases} \tag{4}$$

It is evident that both operators (1) and (2) are right-divided by any operator $\partial - y_0^{-1} y_0'$, where y_0 is a particular solution. For example, they are divided by the singular operator $\partial - x^2 \coth(x^3/3)$ or by nonsingular one $\partial - x^2 \tanh(x^3/3)$; both are not polynomial in x. However the following detail is more interesting: when we consider two of such operators of the first order corresponding to different y_0, then their right nonsingular least common multiple is an operator of the fourth (!) order.

In some cases the elimination process may be completed shortly, for example when the first equation is $y_1' = 0$. The next two examples illustrate other interesting phenomena. The first one is the following system:

$$\begin{cases} y' = \varphi z, \\ z' = \varphi y, \quad \varphi = \exp(-1/x^2). \end{cases}$$

Here the unknown y satisfies an irregular singular equation of the second order and there are no nonsingular equations for y. The second system looks like this:

$$\begin{cases} y' = \psi z, \\ z' = \psi y, \quad \psi = \exp(-\varphi^{-1}). \end{cases}$$

In this case the equation for y has an unremovable exponential singularity. This occurrence seems surprising. The component y in the general solution of a regular linear system of order n can satisfy only strongly degenerate equations. Hence the degenerate equations have only smooth solutions, forming at least an n-dimensional space. In fact this question is not yet analyzed in details, see Proposition 3 below. The problem may be clarified with the help of a few quite easy statements in terms of the differential algebra. The following proposition solves the problem of equivalence of systems and equations.

Proposition 1 *Any nonsingular (nonlinear) system $y' = f(x, y)$ of order n is equivalent (in the nonlinear case only locally) to the equation of the same order for the unknown $z = \exp(\lambda_1 x) y_1 + \ldots + \exp(\lambda_n x) y_n$ for almost all λ_i or for the unknown $u = y_1 + \lambda_1 x y_2 + \lambda_2 x^2 y_3 + \ldots$ for almost all λ_i, $i = 1, \ldots, n$.*

Proof One can differentiate n times the unknown z, then by the implicit function theorem determine the variables $y_1, \ldots, y_n$ from all other equations excluding the last one, and finally substitute y_k in the n-th relation.

The principle part of the jacobian of the obtained expression is equal to the Vandermonde determinant of λ. As a result all the unknowns y_k can be expressed via z and its derivatives; hence the equation for z will be equivalent to the system. The case of the unknown u is considered in the same manner. Of course this procedure of the elimination is not the only one possible.

3 Elimination process

Let us analyze the case of systems of linear differential equations in more details.

Proposition 2 *A nonsingular equation in a neighborhood of the origin of x for an unknown y_1 of an analytic linear system of differential equations $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$ of order n can be obtained by finite number of differentiations of the first equation $y_1' = f_1(x, \mathbf{y})$ and by substitutions of the corresponding derivatives occurring in the system. The order of the regular equation can be inconsistent with the order n of the system. The order of the minimal singular equation will not exceed n.*

Proof Let $c_2, \ldots, c_n$ be the first integrals depending linearly on the unknowns $\mathbf{y}$. For example, one can take the Cauchy data for $x = 0$. Then in the coordinate system $(y_1, c_2, \ldots, c_n)$ the system in question looks like this:

$$\begin{cases} y_1' = a_1 y_1 + a_2 c_2 + \ldots + a_n c_n, \\ c_2' = \ldots = c_n' = 0. \end{cases} \tag{5}$$

Then differentiation and substitutions of derivatives are reduced (under the condition $\mathbf{c} = \text{const}$) to the partial differentiation of x. From the first equation one obtains

$$y_1^{(k+1)} = (a_1 y_1)^{(k)} + a_2^{(k)} c_2 + \ldots + a_n^{(k)} c_n.$$

Clearly one can differentiate the equation once and again until the rank of the matrix $(a_l^{(k)}(0))$ will be stabilized. Then differentiate once again for the last time. We can substitute into the last equation

the expressions of the integrals $c_2, \ldots, c_n$, obtained from the minor of maximal rank of the preceding equations. In a different way, one can eliminate the unknown $c_2, \ldots, c_n$, one after the other. In order to get the minimal singular equation one has to write down the first linear relation over the ring of analytic functions between the rows $(a_j), (a_j'), (a_j''), \ldots$, where $j > 1$. It is clear, that this relation appears not later than we reach the row $(a_j^{(n)})$. Hence, the order of the obtained equation will not exceed n. It turns out that the singular equations obtained from systems of differential equations have the very particular properties.

Proposition 3 *Suppose that the coefficients of the system are smooth (or analytic) from the right side of the origin. Then all the solutions of the minimal singular equation from Proposition* 2 *will be smooth from the right, including the origin. In the analytic case they will be analytic in the origin.*

Proof Let the rows $(a_j), (a_j'), \ldots, (a_j^{(k)})$, $j > 1$, be linearly depended over the ring of analytic functions. Suppose that $\det\left(a_j^{(i)}\right)$, $2 \leq j \leq k+1$, $0 \leq i \leq k-1$, doesn't vanish identically for $x > 0$. Then Wronski's theorem implies that the functions $a_2, \ldots, a_{k+1}$ are linearly independent over the base field while the remaining functions $a_{k+2}, \ldots, a_n$ will depend (linearly, over the field) on the first ones. Hence the number of arbitrary constants of the system (5) will decrease up to k. Further there is one-to-one correspondence between solutions of the singular equation and of the corresponding system of differential equations in a neighborhood of any point $x_0 > 0$ by means of the juxtaposition of initial data at this point. Hence, any solution y_1 of the equation will be a solution of the system and can be smoothly extended to the origin. In the analytic case it can be prolonged analytically.

In the case when the system is singular, that is, it can be represented as $x^m \mathbf{y}' = \mathbf{f}(x, \mathbf{y})$, where $\mathbf{f}$ is analytic, then the above considerations don't work. However, in this case we don't need to control singularities. Of course, one can obtain equations with different orders of singularities.

In the particular case of nonsingular holonomic systems the question is reduced to the discussed case because such a system splits into systems of ordinary differential equations in every independent variable.

Let us consider singular holonomic systems. Again we are interested in an ordinary differential equation of a fixed independent variable which is satisfied by a given unknown. All remaining independent variables play the role of parameters. Thus, we have the following system

$$\begin{cases} \Delta y' = ay + cz, \\ \Delta z' = by + dz. \end{cases}$$

Here Δ is an non-invertible factor, all the coefficients depend on x and on parameters, y is a scalar, and z is a vector-column. Of course, instead of $\Delta\partial$ one can consider an arbitrary vector field multiplied by the function Δ.

Proposition 4 *The unknown y of the above system satisfies an equation containing the top derivative multiplied by a power of Δ.*

Proof One can differentiate the first scalar equation by virtue of the second (vector) one and get

$$\Delta^k y^{(k)} + P_k[y] = c_k z.$$

The degree of the operator P_k will be less than k. The analytic module generated by vectors c_k has a finite number of generators. When we reach the last one we will be able to eliminate the unknown z in the next equation. This procedure doesn't touch the top derivative of y in the last equation.

4 Holonomic D-modules

It is possible to clarify some aspects of the elimination method together with the equivalence problem in the context of the D-modules theory with the help of differential algebra. Let us recall some basic notions and facts (see [12, 7]).

First let us consider the simplest case of a nonsingular holonomic linear homogeneous system given in normal form. In this case derivatives of unknowns of all variables are expressed linearly via the functions itself. A differential module can be associated with such a system in the following way. Let A be the ring of functions of independent variables $x_1, x_2, \ldots$, and let D be the ring of differential operators in $x_1, x_2, \ldots$ with coefficients in A. Let us consider a free A-module M generated by $y_1, y_2, \ldots$, identified with unknowns of the system in question; M is endowed with the structure of a left module over

D. Derivatives of functions from A are well-defined while derivatives of the variables $y_1, y_2, \ldots$ are determined by equations of the system. For example,

$$\frac{\partial y_3}{\partial x_2} = \sum_k a_{32k}(x_\cdot) y_k .$$

The A-module and D-module structures on M are compatible in virtue of the canonical inclusion $A \subset D$. Thus, the D-module M is a desirable differential module. One can consider the ring A consisting of regular functions of various types: smooth, real-analytic, holomorphic functions or polynomials. In the same way it is also possible to consider the corresponding quotient rings or rings of germs of such functions as well as rings of formal power or fractional power series, etc. In fact, the choice of A is completely defined by a problem under investigation. For example, in studying of holonomic systems of differential equations (that is, maximally overdetermined systems) with singularities it may be convenient to make possible division by coefficients of derivatives in order to express the derivatives via variables itself, or, in other words, to consider the localization on a multiplicative set containing all coefficients of the derivatives. In the case of inhomogeneous systems of similar type the A-dimension of the corresponding module M should be increased by one on account of the direct summand A when the right side of the system in question is contained in it. Otherwise, one should add an appropriate D-module containing all the elements of the right side.

Let B be a left D-module, and let $Y_\cdot \in B$ be a solution of the system. Then it is possible to interpret this solution as a homomorphism of D-modules $M \longrightarrow B$ which extends the correspondence $y_n \longmapsto Y_n$ transferring the unknowns $y_\cdot$ to the given solutions $Y_\cdot$. By definition, the image $y_\cdot$ under any D-homomorphism will be a solution in the image of the module M. Thus, M contains a solution $y_\cdot$ of the system trivially.

The set B might coincide with either A or some functional space such as distributions, the completion of A by logarithms or exponents of its elements, by functions depending not only on $x_\cdot$ but on parameters $C_\cdot$ which are arbitrary constants (this space contains fundamental solutions), and so on.

A set of differentiations generating the ring D is not canonically defined. It may coincide or not with all K-differentiation of the algebra A over the base field or the ring K. For example, in case of systems with singularities one can avoid the localization by taking differentiations,

which are degenerated in singular points, as A-generators of the A-algebra D. Thus, it is often more convenient to use the Pochhammer derivative xd/dx and $D = A[xd/dx]$ than the usual one d/dx (see [10], [5]). Set $z = a_1 y_1 + a_2 y_2 + \ldots \in M$, and let

$$\mathrm{Ann}_D z = (0_M : z) = \{p \in D : pz = p[z] = 0\} \subset D$$

be the annihilator of z, a left ideal in D.

Proposition 5 *The set of equations which are satisfied by the variable z by virtue of the given system coincides with $\mathrm{Ann}_D z[z] = 0$. (It seems more reasonable to use the notations $\mathrm{Ann}_D z[Z] = 0$, where $z \in M$ and Z is an unknown.) In other words the operators of these equations generate the left ideal $\mathrm{Ann}_D z$ in the ring D.*

As we have already seen the annihilator of z can be equal to zero. In this case the equation $0 = 0$ is the only one satisfied by z.

Proposition 6 *Let A be the ring of regular functions. The unknown z satisfies (in virtue of the system) a nonsingular holonomic system of differential equations (with the only unknown z) if and only if the submodule $Dz \subset M$ is finitely generated over A.*

In fact, in such a case the derivatives of z of large enough order with respect to all the variables are expressed via generators of the submodule. One can combine these expressions in a suitable way in order to get the minimal holonomic system.

Proposition 7 *The holonomic system satisfied by the unknown z is equivalent to the input system with unknowns $y_.$ if and only if $Dz = M$. It is clear that in such a case the isomorphism $M \simeq D/\mathrm{Ann}_D z$ holds.*

In fact, both conditions mean that the unknown z is expressed via the unknowns y_n while the latter are expressed via the unknown z again.

Let us study relations between elements generating the module over the ring D in case when at least one such an element exists. Let $M = Dz$. Consider the case of systems of ordinary differential equations with constant coefficients. Then one can choose an unknown z in such a way that $\mathrm{Ann}_D z$ is a principal ideal generated by the characteristic polynomial of $\xi = d/dx$ of the system. However, the following simplest example shows that the problem is more complicated. Let $M = Ay_1 \bigoplus_A Ay_2$, $y_1' = 0$, $y_2' = 0$. Then $M \neq Dy_1$,

$M \neq Dy_2$, but $M = Dz = Dz_1 = Dz_2$ for $z = y_1 + xy_2$, $z_1 = y_1 + e^x y_2$, $z_2 = y_1 + \sin x\ y_2$, and the corresponding annihilators are generated by operators of the following equations: $z'' = 0$, $z_1'' = z_1'$, $z_2'' = -\tan x\ z_2'$. In the general situation set $M = Dz = Dw$. Then $z = pw$, $w = qz$, for some $p, q \in D$. Consequently, $z = pqz$, $w = qpw$, $\mathrm{Ann}_D z\ p \subset \mathrm{Ann}_D w$, $\mathrm{Ann}_D w\ q \subset \mathrm{Ann}_D z$. And conversely:

Proposition 8 *Let $M = Dz$, $z = pqz$ for some $p, q \in D$, $w = qz$. Then $M = Dw$, $\mathrm{Ann}_D w = \mathrm{Ann}_D z :_D q$ $(= \{p; pq \in \mathrm{Ann}_D z\})$, $\mathrm{Ann}_D z = \mathrm{Ann}_D w :_D p$.*

In the case of one independent variable the annihilators of generators are principal ideals; then it is possible to define relations between them more precisely.

Corollary 1 *Let $M = Dz = Dw$, $z = pw$, $w = qz$, $p, q \in D$, and $\mathrm{Ann}_D z$, $\mathrm{Ann}_D w$ be principal ideals: $\mathrm{Ann}_D z = Dp_0$, $\mathrm{Ann}_D w = Dq_0$. Then $p_0 p = p_1 q_0$ and $q_0 q = q_1 p_0$ for some $p_1, q_1 \in D$. In other words, the operators p_0 and q_0 are equivalent.*

The main corollary is the following.

Corollary 2 *The set of all unordered couples $\{p, q\}$ of operators such that $rz = pqrz$, $M = Drz$, for some $r \in D$, doesn't depend on the choice of a generator z. Hence, it is a characteristic of the module M.*

In the classical theory the meaning of this set is not very clear. In commutative algebra simple annihilators are called associated ideals. We don't know whether it is possible to extend this notion to the non-commutative or differential cases. In fact, in the differential algebra associated ideals over A are trivial for non-degenerate systems.

The above considerations from differential algebra solve the problem of reformulation of main questions concerning systems of differential equations satisfied by the versal integrals; they will be discussed in the next sections. For completeness let us also describe differential objects which naturally arise in the general situation.

The system of differential equations $f_\cdot(x_\cdot, \partial^\cdot y_\cdot) = 0$ is usually described in terms of independent variables $x_\cdot$ and dependent many-dimensional variables $y_\cdot$. Here $\partial^\cdot$ denotes the derivatives of some fixed order of independent variables. Additional parameters of the system above can be identified with either independent variables with respect to which the differentiations are not performed or dependent variables whose all the derivatives are equal to zero. The module G, containing

infinite number of independent variables $x_.$, $y_{.,\dots}$, corresponds to such a system. Here the variables $y_{.,\dots}$ denote the derivatives of the functions y_n; the multi-index $\dots$, following after the comma enumerates all the variables x_k, by which the variable y_n will be differentiated when it will become a function. The variables $x_.$ are called coordinates while $y_{.,\dots}$ are called unknowns. The module G is endowed by the structure of a differential module over the ring of differential operators $D \subset A[D_.]$, where D_k is the mass (in physics terminology) differentiation with respect to the variable x_k so that $D_k x_j = \delta_{kj}$, $D_k\, y_{m,(\alpha)} = y_{m,(\alpha,k)}$. Here the ring A is the same as in the beginning; it is contained in G while elements (functions) in G might be polynomials of $y_.$ in case when the system is polynomial. Otherwise, they form a D-submodule of the module of differentiable functions each of which is depended only on a finite number of all variables under consideration. Such functions are called differential functions by P. Olver [12]. It seems it is not quite correct since they don't contain differentials at all. Following I. Newton [3] such functions should be called fluxional (or "derivatival"?). But in this case differential equations would be renamed as fluxional ones. Left sides of the system must belong to G (when the variables $y_{.,\dots}$ are considered as the corresponding derivatives of the unknown functions $y_.$). The quotient G by the D-submodule generated by these left sides determines the desirable D-module $M = G/(D\,f_.)$.

Two inductive systems are defined in the modules M and G, respectively, consisting of A-submodules which contain functions depending on increasing numbers of variables. There are two filtrations $M_n \subset M_{n+1}$, $G_n \subset G_{n+1}$. A solution of the system is defined as above; it is a D-homomorphism $M \longrightarrow B$. The support of B is mapped into the support of M and into the projective system of supports: $\mathrm{Supp}B \longrightarrow \mathrm{Supp}M_n$. In other words under substitutions the solution transforms all the functions of the submodule $Df_.$ to zero. Due to B. Malgrange [7] the projective system of supports $\mathrm{Supp}M_n$ is called D-manifolds or "diffeofolds" (D-variété, certaines disent "diffiété"). As it has been remarked before it is possible to call them differential manifolds following to P. Olver [12]. Above all, the space of values of all variables is endowed by the structure of fibration given by the projection on the space of values of independent variables. On this fibration mass differentiations define a connection. In this case solutions are obtained as horizontal sections of the connection. (The module B is then the space of such sections). In the case when they are too smooth to be solutions it is possible to find them in the tensor

product of the space of sections and a suitable sheaf of functions.

If the system is linear, the module M contains the D-submodule M_L of functions which are linear in $y_{\ldots,\ldots}$. If the system contains expressions of all higher derivatives via ones of lower orders then M_L is finitely generated over A and coincides with the module M which has been defined at the beginning. Otherwise, one needs to use an infinite number of variables. For instance, in the case of the simplest degenerate equation $f = x\,y' + y = 0$ (one can take also $x\,y' = 0$) if one can not allow to divide by x and doesn't want to decrease the algebra of differentiation then the ideal Df is generated by the functions f, $f' = x\,y'' + 2y'$, $f'' = x\,y''' + 3y''$, $\ldots$, while the module M_L is equal to $\bigoplus\limits_k Ky^{(k)} \bigoplus Ay$, where A is the ring of smooth functions, K is the base field, and the direct sum is taken over K. In the case when a system is not holonomic the module M is A-infinite also. For example, for the simplest (but non ordinary) heat equation $f = u_t - u_{xx} = 0$, one obtains $M_L \cong A\,u \bigoplus\limits_A A\,u_x \bigoplus\limits_A A\,u_{xx} \bigoplus \cdots$ expressing the t-derivatives via x-derivatives.

Of course, the set of differentiations G is not exhaustive by mass derivatives with respect to the independent variables and their combinations. It also contains partial derivatives with respect to all variables. Differentiations of G preserving the submodule of the system Df are called symmetries; they are defined on M. Symmetries are defined not alone but as Lie subalgebras. Their commutators with mass derivatives (which are trivial symmetries) remain symmetries as well as commutators with other symmetries.

There is the question: why is it necessary to add an infinite number of other variables not contained in the system? However, at present there is no other way to consider a system of differential equations as a known algebraic structure. Above all there are some considerations to avoid restrictions on orders of derivatives. For example the higher symmetries referred to higher derivatives may be very useful. A limited number of unnecessary higher derivatives were introduced in mathematical physics in order to transform a non-linear system to a quasi-linear one. It is possible, since after differentiation higher derivatives become cofactors. Because of this, functions from G may be defined as polynomials in higher derivatives, beginning from some fixed order.

A prescribed separation of dependent and independent variables is inconvenient when we are interested in the integral manifolds of the

system as well as its solutions. In order to avoid the inconvenience it is possible to use the language of differential forms. Thus, the heat equation can be interpreted as a d-closed ideal contained in the exterior A-algebra of differential forms in the variables t, x, u, u_t, u_x, u_{tx}, with two generators $\omega_1 = du - u_t dt - u_x dx$, $\omega_2 = du_x - u_{tx} dt - u_t dx$. Here the integral manifold is defined as a two-dimensional manifold on which restrictions of these forms vanish. This manifold will be a solution $u = u(t, x)$ in a neighborhood of the points, where $dt \wedge dx$ doesn't vanish (in virtue of the implicit function theorem). This language is very fruitful in order to transform coordinates, especially in the case when they are functions of the dependent and independent variables simultaneously. At the first sight it seems that in such a case it is possible to use a finite number of unknowns only. However this is not true. In fact, the introduced coordinates and algebra of differential forms belong to a component of the filtered algebra. Acting on forms as Lie derivatives, the mass derivatives realize the outcome of other components. If one drops this action it will be in fact equivalent to the prohibition to differentiate equations of the system.

In conclusion the following detail should be remarked. Although the integrals studied in our work satisfy linear systems of differential equations, non-linear equations can appear in the linear theory. For example, the use of symmetries may give rise to non-linearity. In this connection one can mention a well-known method of passage from a linear equation of second order (in a scalar or matrix form) to the Ricatti equation. In this case the property of linearity is lost, however the order will be doubled. In addition, it is possible to reformulate nonlinear Propositions 2 and 7 in terms of D-modules but we will refrain from it.

5 Local Picard-Fuchs systems and A_μ-singularities

It is well-known that the notion of Gauss-Manin or Picard-Fuchs systems has a natural interpretation in the context of singularity theory. Following [13] let us consider a flat map $f = (f_1, \ldots, f_m) \colon X \to S$ of germs of complex spaces such that the fibre X_0 over the distinguished point $0 \in S$ is an isolated complete intersection singularity with Milnor number μ, and $\dim X_0 = n \geq 1$. Suppose that f is the minimal versal deformation of X_0. Then $X \subseteq U \times S$, where $U \subset \mathbf{C}^m$

is a ball centered in the singular point of X_0, and the discriminant $D \subset S$ of this deformation is a hypersurface without multiple irreducible components. Set $S' = S \setminus D$, $X' = X \setminus f^{-1}(S')$. Then small enough representatives of all the fibers X_s, $s \in S'$, are smooth germs and the $\mathcal{O}_S$-module $\mathcal{H}^n(f_*\Omega^\bullet_{X/S})$ is free of rank $\mu = \dim H^n(X_s, \mathbf{C})$, while $\mathcal{H}^p(f_*\Omega^\bullet_{X/S}) = 0$ for all $0 < p < n$. Let ω be a holomorphic form representing a class of the relative de Rham cohomology in $f^{-1}(B)$, where B is a small ball in S', and let $\gamma(s)$ be any continuous family of cycles of the Milnor fibration over B. Then the integral

$$I(s) = \int_{\gamma(s)} \omega$$

is a holomorphic function on B. Making use of the Stokes' formula one obtains

$$dI = \sum \eta_i ds_i, \quad \eta_i(s) = \int_{\gamma(s)} \omega_i,$$

where $d\omega = \sum df_i \wedge \omega_i$, and ω_i are holomorphic in $f^{-1}(B)$. Hence, the function $I(s)$ is constant if and only if $\eta_i \equiv 0$, for all i. In particular, there is defined a non-singular connection

$$\nabla_{X'/S'} \colon \mathcal{H}^n(f_*\Omega^\bullet_{X'/S'}) \to \mathcal{H}^n(f_*\Omega^\bullet_{X'/S'}) \otimes_{\mathcal{O}_{S'}} \Omega^1_{S'} \tag{6}$$

given as follows:

$$\nabla_{X'/S'}([\omega]) = \sum_{i=1}^{\mu} [\omega_i] \otimes ds_i,$$

where the cohomology class of the differential form ω is denoted by $[\omega]$. This differential operator is called the local Gauss-Manin connexion. Consider holomorphic forms ω_i, $i = 1, \ldots, \mu$, on X' whose restrictions on every fiber X'_s of the map $f' \colon X' \to S'$ generate $H^n(X'_s, \mathbf{C})$. The corresponding sections w_i, $i = 1, \ldots, \mu$, of the bundle $\mathcal{H}^n = \mathcal{H}^n(f_*\Omega^\bullet_{X'/S'})$ induce a basis. By construction, $d\omega_i = \sum df_j \wedge \omega_{ij}$, where ω_{ij} are holomorphic on X', and the corresponding holomorphic sections of the cohomological bundle can be written as $\sum_k \vartheta^k_{ij} w_k$, where ϑ^k_{ij} are holomorphic on S'. One then obtains $d\,\underline{w} = \Theta \underline{w}$, where $\underline{w} = (w_1, \ldots, w_\mu)^T$ is a vector-column and $\Theta = \| \vartheta^k_i \|$ is the matrix with entries $\vartheta^k_i = \sum \vartheta^k_{ij} ds_j$. The integration of the sections w_i along any continuous family of cycles $\gamma_i(s)$ gives us $I_i(s)$. Then the following system of differential equations describes the connection (6) :

$$d\,\mathbf{I} = \Theta \cdot \mathbf{I}, \tag{7}$$

where $\mathbf{I} = (I_1, \ldots, I_\mu)^T$ is a vector-column of unknown functions, Θ is the connection form of the differential operator $\nabla_{X'/S'}$, and all entries of Θ are holomorphic 1-forms on S' satisfying the integrability condition $d\,\Theta = \Theta \wedge \Theta$. Such system is called the local Picard-Fuchs system.

In the case of A_μ-singularities, $\mu \geq 1$, there is a general method for computing the matrix Θ (see [13], [9]). Let $X_0 \subset \mathbf{C}^{\nu+1}$ be a hypersurface with an isolated A_μ-singularity given by the equation $\varphi(z, y) = z^{\mu+1} + y_1^2 + \ldots + y_\nu^2 = 0$. Set $P_\mu(s, z) = z^{\mu+1} + s_{\mu-1} z^{\mu-1} + \ldots + s_1 z + s_0$. Then the minimal versal deformation $X \subseteq \mathbf{C}^{\nu+1} \times S$ of the germ X_0 over the base space $S = \mathbf{C}^\mu$ is determined by the equation

$$F(s, z, y) = P_\mu(s, z) + y_1^2 + \ldots + y_\nu^2 = 0.$$

The projection $f\colon X \to S$ is a smooth locally trivial fibration over the complement of the discriminant D given by the equation $\Delta(s) = 0$, where Δ is the discriminant of the polynomial $P_\mu(s, z)$. The differential forms

$$\omega_\ell = \frac{z^\ell dz \wedge dy_1 \wedge \ldots \wedge dy_\nu}{dF}, \quad \ell = 0, 1, \ldots\,, \tag{8}$$

are holomorphic on $X' = X \setminus f^{-1}(D)$. In other words, ω_ℓ is determined by the relation $\omega_\ell \wedge dF = z^\ell dz \wedge dy_1 \wedge \ldots \wedge dy_\nu$. The first μ elements from this set form a basis in the cohomological bundle. Then the Picard-Fuchs system for the integrals

$$I_\ell = \int_{\gamma(s)} \omega_\ell = \int_{\gamma(s)} \frac{z^\ell dz \wedge dy_1 \wedge \ldots \wedge dy_\nu}{dF}, \tag{9}$$

can be described as follows. First, let us remark that the system contains the following equations:

$$\frac{\partial I_i}{\partial s_j} = \frac{\partial I_p}{\partial s_q}, \quad i + j = p + q, \quad 0 \leq i, j, p, q \leq \mu - 1, \tag{10}$$

which we call the cyclic relations. It is possible to obtain these relations in the following way. Set

$$G(s, z, y) = F(s, z, y) - y_\nu^2 = P_\mu(s, z) + y_1^2 + \ldots + y_{\nu-1}^2.$$

Making use of the substitution $y_\nu = \sqrt{F - G}$ for the unknowns y_ν and F, one obtains

$$dz \wedge dy' \wedge dy_\nu = \frac{1}{2} \frac{dz \wedge dy' \wedge dF}{\sqrt{F - G}},$$

where $dy' = dy_1 \wedge \cdots \wedge dy_{\nu-1}$. Taking $F = 0$ one gets

$$\omega_0 = -\frac{i}{2}\frac{dz \wedge dy'}{\sqrt{G}}, \quad \omega_\ell = z^\ell \omega_0,$$

where $i = \sqrt{-1}$. As a result, we obtain the following representation

$$I_\ell = -\frac{i}{2}\int_{\gamma(s)} z^\ell G^{-1/2} dz \wedge dy', \quad \ell = 0, 1, \ldots . \tag{11}$$

Let us introduce the potential in the following way:

$$I = -i\int_{\gamma(s)} G^{1/2} dz \wedge dy'.$$

Computing the first and second partial derivatives of the potential we get the following equalities

$$I_\ell = \frac{\partial I}{\partial s_\ell} = -\frac{i}{2}\int_{\gamma(s)} z^\ell G^{-1/2} dz \wedge dy', \quad 0 \le \ell \le \mu - 1,$$

as well as all the cyclic relations (10) :

$$\frac{\partial^2 I}{\partial s_i \partial s_j} = \frac{\partial I_i}{\partial s_j} = \frac{\partial I_p}{\partial s_q} = \frac{i}{4}\int_{\gamma(s)} z^{i+j} G^{-3/2} dz \wedge dy',$$

where $i + j = p + q, \quad 0 \le i, j, p, q \le \mu - 1$.

For convenience of representation of the other equations of the desired Picard-Fuchs system one should consider $\mu + 2$ integrals, I_0, $I_1, \ldots, I_{\mu+1}$. Let us denote

$$J_k = \frac{\partial I_i}{\partial s_j}, \quad i + j = k, \quad k = 0, 1, \ldots, 2\mu,$$

and let $\mathbf{J}$ be a vector-column generated by the functions J_k.

Proposition 9 *The rest of the Picard-Fuchs system for the integrals* (11) *can be written as follows*

$$\Theta \cdot \mathbf{J} = \mathbf{I}',$$

where Θ *is the resultant matrix of the polynomials* $z^{\mu+1} P_\mu(s, z^{-1})$ *and* $z^\mu \frac{\partial P_\mu}{\partial z}(s, z^{-1})$, *while* $\mathbf{I}'$ *is the vector-column,*

$$\mathbf{I}' = ((\tfrac{\nu}{2} - 1)I_0, \ldots, (\tfrac{\nu}{2} - 1)I_{\mu-1}, 0, -I_0, -2I_1, \ldots, -\mu I_{\mu-1})^T.$$

Proof First, multiply the identity $P_\mu = G - (y_1^2 + \ldots + y_{\nu-1}^2)$ by

$$\frac{i}{4} z^k G^{-3/2} dz \wedge dy', \quad k = 0, \ldots \mu - 1.$$

Then making use of integration one gets

$$J_{\mu+1+k} + s_{\mu-1} J_{\mu-1+k} + \ldots + s_0 J_k = \\ - \frac{1}{2} I_k - \frac{i}{4} \int_{\gamma(s)} z^k (y')^2 G^{-3/2} dz \wedge dy' = \frac{\nu - 2}{2} I_k,$$

because the integral is equal to

$$-\frac{i}{4} \sum_{j=1}^{\nu-1} \int_{\gamma(s)} z^k y_j^2 G^{-3/2} dz \wedge dy' = \\ \frac{i}{4} \sum_{j=1}^{\nu-1} \int_{\gamma(s)} z^k y_j dz \wedge \ldots dy_{j-1} \wedge dG^{-1/2} \wedge dy_{j+1} \wedge \ldots,$$

while making use of integration by parts the latter integral can be reduced to the following form:

$$-\frac{i}{4} \sum_{j=1}^{\nu-1} \int_{\gamma(s)} z^k G^{-1/2} dz \wedge dy' = \frac{\nu - 1}{2} I_k.$$

The other $\mu + 1$ equations one can obtain by transformation of the integrals $J_{\mu+k}$, $k = 0, \ldots, \mu$, in the following way:

$$(\mu + 1) J_{\mu+k} = \\ \frac{i}{4} \int_{\gamma(s)} z^k G^{-3/2} d(z^{\mu+1}) \wedge dy' = \frac{i}{4} \int_{\gamma(s)} z^k G^{-3/2} dG \wedge dy' - \\ \frac{i}{4} \int_{\gamma(s)} z^k G^{-3/2} d(s_{\mu-1} z^{\mu-1} + \ldots + s_0) \wedge dy' = \\ -k I_{k-1} - (\mu - 1) s_{\mu-1} J_{\mu-2+k} - \ldots - s_1 J_k.$$

This completes the proof.

It is not difficult to see that Θ is a square matrix of order $2\mu + 1$ consisting of μ rows $(s_0, s_1, \ldots, s_{\mu-1}, 0, 1, 0, \ldots)$ and $\mu + 1$ additional rows $(s_1, 2s_2, \ldots, (\mu - 1)s_{\mu-1}, 0, \mu, 0, \ldots)$ each shifted one position to the right in relative to the preceding one (see below the explicit expressions for $\mu = 2, 3$ in (12) and (14), respectively, while the general case is written down in the next section, equations (21)-(22)).

The determinant of Θ is the discriminant of the polynomial $P_\mu(s,z)$ so that the Picard-Fuchs system over $S \setminus D$ is the system in total differentials. In other words the characteristic set $M \subset T^*(S)$ of this system is concentrated over the discriminant D. Further, its dimension is equal to μ, that is, the Picard-Fuchs system is a maximally overdetermined or, equivalently holonomic system of differential equations.

For example, let consider the case of the versal deformation of an A_2-singularity, that is, the two-parameter family given by $P_2(s,z) = z^3 + s_1 z + s_0$. Then the Picard-Fuchs system for the seven integrals $I_0, I_1, J_0, \ldots, J_4$ can be represented in the following form:

$$\begin{array}{rcl} s_0 J_0 + s_1 J_1 + J_3 &=& (\nu/2-1) I_0, \\ s_0 J_1 + s_1 J_2 + J_4 &=& (\nu/2-1) I_1, \\ s_1 J_0 + 3 J_2 &=& 0, \\ s_1 J_1 + 3 J_3 &=& -I_0, \\ s_1 J_2 + 3 J_4 &=& -2 I_1, \end{array} \tag{12}$$

Eliminating $J_2, \ldots, J_4$ one gets

$$\begin{array}{rcl} s_0 J_0 + \frac{2}{3} s_1 J_1 &=& (\frac{\nu}{2} - \frac{1}{6}) I_0, \\ -\frac{2}{9} s_1^2 J_0 + s_0 J_1 &=& (\frac{\nu}{2} + \frac{1}{6}) I_1. \end{array} \tag{13}$$

The determinant of the coefficient matrix of the left side of the system is equal to $s_0^2 + \frac{4}{27} s_1^3$, the discriminant of $P_2(s,z)$. The next example is the versal deformation of an A_3-singularity. In this case we have the 3-parameter family given by $P_3(s,z) = z^4 + s_2 z^2 + s_1 z + s_0$; the Picard-Fuchs system for the integrals $I_0, I_1, I_2, J_0, \ldots, J_6$, looks like this:

$$\begin{array}{rcl} s_0 J_0 + s_1 J_1 + s_2 J_2 + J_4 &=& (\nu/2-1) I_0, \\ s_0 J_1 + s_1 J_2 + s_2 J_3 + J_5 &=& (\nu/2-1) I_1, \\ s_0 J_2 + s_1 J_3 + s_2 J_4 + J_6 &=& (\nu/2-1) I_2, \\ s_1 J_0 + 2 s_2 J_1 + 4 J_3 &=& 0, \\ s_1 J_1 + 2 s_2 J_2 + 4 J_4 &=& -I_0, \\ s_1 J_2 + 2 s_2 J_3 + 4 J_5 &=& -2 I_1, \\ s_1 J_3 + 2 s_2 J_4 + 4 J_6 &=& -3 I_2. \end{array} \tag{14}$$

Eliminating $J_3, \ldots, J_6$ one gets

$$\begin{array}{rcl} s_0 J_0 + \frac{3}{4} s_1 J_1 + \frac{1}{2} s_2 J_2 &=& (\frac{\nu}{2} - \frac{3}{4}) I_0, \\ -\frac{1}{8} s_1 s_2 J_0 + (s_0 - \frac{1}{4} s_2^2) J_1 + \frac{3}{4} s_1 J_2 &=& (\frac{\nu}{2} - \frac{1}{2}) I_1, \\ -\frac{3}{16} s_1^2 J_0 - \frac{1}{2} s_1 s_2 J_1 + (s_0 - \frac{1}{4} s_2^2) J_2 &=& (\frac{\nu}{2} - \frac{1}{4}) I_2 + \frac{1}{8} s_2 I_0. \end{array} \tag{15}$$

The determinant of the coefficient matrix of the left side of the system relative to J_0, J_1, J_2 is equal to the discriminant of $P_3(s,z)$:

$$\Delta = s_0^3 - \tfrac{1}{2}s_0^2 s_2^2 + \tfrac{1}{2^4}s_0 s_2^4 + \tfrac{3^2}{2^4}s_0 s_1^2 s_2 - \tfrac{1}{2^6}s_1^2 s_2^3 - \tfrac{3^3}{2^8}s_1^4. \qquad (16)$$

Let also remark that the differential forms (8) can be presented in the case $\nu = 1$ as follows

$$\omega_\ell = \frac{z^\ell dz \wedge dy_1}{dF} = -\frac{i}{2}\frac{z^\ell dz}{\sqrt{P_\mu(s,z)}}, \quad \ell = 0, 1, \dots,$$

where $i = \sqrt{-1}$. Hence, the integrals (9) and (11)

$$I_\ell(s) = -\frac{i}{2}\int_\gamma \frac{z^\ell dz}{\sqrt{P_\mu(s,z)}}, \quad \ell = 0, 1, \dots, \mu+1, \qquad (17)$$

are well-defined for any continuous family of cycles in the Milnor fibration of the versal family over $S \setminus D$. Further, these integrals are μ-valued analytic functions branched over D; they satisfy the Picard-Fuchs system just described.

6 Elliptic, hyperelliptic and versal integrals

A clear illustration of the above material and its relations with the theory of versal integrals is contained in the classical theory of elliptic and hyperelliptic integrals (see [14]). To be more precise, let $P = P(t,z)$ be a unitary polynomial in one variable z of degree $n+1 \geq 2$ whose coefficients are polynomials in $t = (t_1, \dots, t_m)$. Consider the following integral

$$I(t) = \int_\gamma \frac{dz}{\sqrt{P(t,z)}},$$

where $\gamma = \gamma(t)$ is an analytic path in a domain of the Riemann surface of $\sqrt{P(t,z)}$. If t changes in such a way that the roots of the polynomial P don't meet the path γ then the integral defines an analytic function of t. In the case when γ is a cycle on the Riemann surface of $\sqrt{P(t,z)}$ then it is possible to extend analytically the integral along any path which doesn't intersect the hypersurface $\Delta(t) = 0$, the discriminant of P. The corresponding analytic function is, in fact, a multi-valued function on the complement $U = \mathbf{C}^m \setminus \{\Delta(t) = 0\}$. Let now γ be a

cycle on the Riemann surface of $\sqrt{P}$. Then the following integrals are well defined in a similar way

$$I_\ell(t) = \int_\gamma \frac{z^\ell dz}{\sqrt{P(t,z)}}, \quad \ell = 0, 1, \ldots, n-1, \tag{18}$$

so that $I_0(t) = I(t)$. It is possible to differentiate these integrals in the usual way

$$\partial_i I_0(t) = -\frac{1}{2}\int_\gamma \frac{\partial_i P(t,z)}{P(t,z)^{3/2}} dz,$$

where $\partial_i = \partial/\partial t_i,\ i = 1, \ldots, m$. It is not difficult to verify (see [14]) that these integrals satisfy the following linear system of differential equations

$$\partial_i I_j(t) = \sum_{\ell=0}^{n-1} A_{ij}^\ell I_\ell(t), \tag{19}$$

where $j = 0, 1, \ldots, n-1$, and A_{ij}^ℓ are rational functions of t with poles along the discriminant. Set $n = 2$, $m = 1$, and $P = z^3 + s_1 z + s_0$; it is the versal deformation of an A_2-singularity. Then the system (19) for two integrals $I_0(s)$, $I_1(s)$ looks like this (cf. [14], 1.3.2, in a slightly different notations):

$$\begin{aligned}
\partial_0 I_0 &= -\tfrac{9}{2\Delta} s_0 I_0 - \tfrac{3}{\Delta} s_1 I_1, \\
\partial_1 I_0 &= -\tfrac{1}{\Delta} s_1^2 I_0 + \tfrac{9}{2\Delta} s_0 I_1, \\
\partial_0 I_1 &= -\tfrac{1}{\Delta} s_1^2 I_0 + \tfrac{9}{2\Delta} s_0 I_1, \\
\partial_1 I_1 &= \tfrac{3}{2\Delta} s_0 s_1 I_0 + \tfrac{1}{\Delta} s_1^2 I_1,
\end{aligned}$$

where $\Delta = 27 s_0^2 + 4 s_1^3$ is the discriminant of P. Here the second and third equations coincide: it is nothing but a consequence of the cyclic relation $\partial_1 I_0 = \partial_0 I_1$.

The following well-known example appears in the classical theory of elliptic integrals. Set $P = z(z-1)(z-t)$. Then the discriminant of P is equal to $t^2(t-1)^2$. Set $\Delta = t(t-1)$.

Proposition 10 *The elliptic integrals I_0 and I_1 satisfy the following system of differential equations:*

$$\begin{aligned}
\partial_t I_0(t) &= -\tfrac{1}{2\Delta} t I_0(t) + \tfrac{1}{2\Delta} I_1(t), \\
\partial_t I_1(t) &= -\tfrac{1}{2\Delta} t I_0(t) + \tfrac{1}{2\Delta} t I_1(t).
\end{aligned}$$

Proof Instead of the well-known computational algorithm described in [14] we apply a slightly different method which essentially simplifies calculations. The idea of this approach lies in the fact that all the integrand expressions can be reduced to the common denominator equal to $P(t,z)^{3/2}$, and not $P(t,z)^{1/2}$. Making use of integration by parts and the following two identities

$$z^3 = P + (t+1)z^2 - tz, \quad 3z^2 = P' + 2(t+1)z - t,$$

where $P' = \partial P/\partial z$, one gets:

$$I_0(t) = \int_\gamma \frac{dz}{\sqrt{P(t,z)}} = \frac{1}{2}\int_\gamma \frac{zP'dz}{P(t,z)^{3/2}}$$

$$= \frac{1}{2}\int_\gamma \frac{(3P + \frac{1}{3}(t+1)P' + \frac{2}{3}(t+1)^2 z - 2tz - \frac{1}{3}t(t+1))dz}{P(t,z)^{3/2}}.$$

As a result,

$$I_0(t) = -\frac{1}{3}\int_\gamma \frac{((2t^2 - 2t + 2)z - t(t+1))dz}{P(t,z)^{3/2}}.$$

Analogously,

$$I_1(t) = \frac{1}{3}\int_\gamma \frac{(2t^2 - t(t+1)z)dz}{P(t,z)^{3/2}}.$$

By similar computations one obtains the first derivative of I_0 :

$$I_0'(t) = \frac{1}{2}\int_\gamma \frac{(z^2 - z)dz}{P(t,z)^{3/2}} = \frac{1}{6}\int_\gamma \frac{((2t-1)z - t)dz}{P(t,z)^{3/2}}.$$

Analogously,

$$I_1'(t) = \frac{1}{6}\int_\gamma \frac{((t-2)z + t)dz}{P(t,z)^{3/2}}.$$

In conclusion, it remains to solve the following system of linear equations

$$I_0' = aI_0 + bI_1, \quad I_1' = cI_0 + dI_1,$$

for the unknown functions a, b, c, d, depending on t.

The standard elimination process reduces the system just obtained to the Gauss-Legendre hypergeometric equation

$$(t(1-t)\partial_t^2 + (1-2t)\partial_t - \frac{1}{4})I_0(t) = 0. \tag{20}$$

In fact, it is the *minimal* equation (cf. Proposition 4). For completeness we also mention here the following well-known representation for the integral $I_0(t)$ (see [14]):

$$\frac{1}{\pi}\int_1^\infty \frac{dz}{\sqrt{z(z-1)(z-t)}} = {}_2F_1(\frac{1}{2},\frac{1}{2},1;t)\ ,$$

where

$${}_2F_1(a,b,c;t) = 1 + \frac{ab}{c}t + \frac{a(a+1)b(b+1)}{2!c(c+1)}t^2 + \dots$$

is the hypergeometric series.

It should be remarked that there is no confusion in the above notations for the two groups (17) and (18) of integrals $I_0,\ I_1, \dots$, with two distinct contours of integration. In fact, it is possible to show that both groups of integrals satisfy the same Picard-Fuchs system. First recall, following V.P. Palamodov, that the versal integrals are defined as follows

$$I^\lambda(s) = \int_\gamma F(s,z,y)^\lambda dz \wedge dy,$$

where $F(s,z,y) = P_\mu(s,z)+y_1^2+\ldots+y_\nu^2$ is the versal deformation of the isolated hypersurface singularity given by the polynomial $F(0,z,y)$, the contour $\gamma \subset \{(z,y) \in \mathbf{C}^{\nu+1} \colon F(s,z,y) \neq 0\}$ is homotopically equivalent to $\mathbf{R}^{\nu+1}$, and $\lambda \in \mathbf{C}$. In particular, the integral I_0 from (18) can be considered as a versal integral for an A_2-singularity: set $\lambda = -1/2$, $\nu = 1$, and $\mu = 2$. Let us set

$$I_i^\lambda(s) = \int_{\mathbf{R}^n} z^i F(s,z,y)^\lambda dz \wedge dy, \quad i = 0,1,\dots,\mu+1,$$

where $n = \nu + 1$. We now verify that the integrals $I_0,\dots,I_{\mu+1}$ from (9) and the versal integrals $I_0^\lambda,\dots,I_{\mu+1}^\lambda$ satisfy the same Picard-Fuchs system. For simplicity we consider in detail the case of cyclic relations for A_μ-singularities only. Of course the general case can be considered in a similar way. The following proposition and an idea of its proof was explained to the authors by V.P. Palamodov in a seminar at Moscow State University.

Proposition 11 *The integrals $I_0,\dots,I_{\mu+1}$ and $I_0^\lambda,\dots,I_{\mu+1}^\lambda$ satisfy the same cyclic relations, that is,*

$$\frac{\partial I_i}{\partial s_j} = \frac{\partial I_p}{\partial s_q}, \quad \frac{\partial I_i^\lambda}{\partial s_j} = \frac{\partial I_p^\lambda}{\partial s_q}, \quad i+j = p+q, \quad 0 \le i,j,p,q \le \mu-1.$$

Proof Consider the hypersurface $F(s,z,y) = 0$ as a real manifold. Then

$$I_i^\lambda(s) = \int_{\mathbf{R}^n} z^i F(s,z,y)^\lambda dz \wedge dy = \int_{\mathbf{R}^{n-1}} \int_{F=\sigma} \frac{z^i dz \wedge dy}{dF} \sigma^\lambda d\sigma.$$

Denote $(s_0 - \sigma, s_1, \ldots, s_{\mu-1})$ by $s - \sigma$. Then it is possible to rewrite the integral on the right side in the following form:

$$\int_{\mathbf{R}} I_i(s-\sigma)\sigma^\lambda d\sigma = I_i * \sigma^\lambda,$$

where the operator of the convolution is denoted by the star. Hence, one gets

$$\partial_j I_i^\lambda = \lambda \int_{\mathbf{R}^n} z^{i+j} F(s,z,y)^{\lambda-1} dz \wedge dy =$$

$$\partial_0 \int_{\mathbf{R}^n} z^{i+j} F(s,z,y)^\lambda dz \wedge dy = I_{i+j} * \lambda\sigma^{\lambda-1}.$$

It means that the result depends on the sum $i+j$ only and, consequently,

$$\partial_j I_i * \sigma^\lambda = \partial_j I_i^\lambda = I_{i+j} * \lambda\sigma^{\lambda-1} = \partial_q I_p^\lambda = \partial_q I_p * \sigma^\lambda.$$

The desired statement follows immediately since one can eliminate the convolution σ^λ from the integrals on the left and right sides.

Let us write down the Picard-Fuchs system in two particular cases of an A_μ-singularity. At first recall that the system (7) can be presented in the following form

$$s_0 J_p + s_1 J_{p+1} + \ldots + s_{\mu-1} J_{p+\mu-1} + J_{p+\mu+1} + (1-\nu/2) I_p = 0, \quad p = 0, 1, \ldots, \mu-1, \quad (21)$$

$$s_1 J_p + 2s_2 J_{p+1} + \ldots + (\mu-1) J_{p+\mu-1} + (\mu+1) J_{p+\mu} + p I_{p-1} = 0, \quad p = 0, 1, \ldots, \mu. \quad (22)$$

As an easy exercise one can eliminate the unknowns $J_{2\mu-1}$ and $J_{2\mu}$ from the last two equations of the first group (21) of relations with the help of the last two equations of the second group (22). In fact, it gives the system for versal integrals described in [9]. Making use of cyclic relations (10) one can reduce the obtained two equations to the following form

$$((\mu+1)s_0\partial_0 + \mu s_1\partial_1 + \ldots + 2s_{\mu-1}\partial_{\mu-1} - \rho + q) I_{\mu-q} = 0, \quad q = 1, 2,$$

where $\rho = (\mu+1)\nu/2$. Making use of relations (10) once again, the remaining equations can be rewritten as follows:

$$(s_0\partial_0 + \ldots + s_{\mu-1}\partial_{\mu-1})I_p + \partial_{\mu-1}I_{p+2} + (1-\nu/2)I_p = 0, \quad p = 0, 1, \ldots, \mu-3,$$
$$(s_1\partial_0 + 2s_2\partial_1 + \ldots + (\mu-1)s_{\mu-1}\partial_{\mu-2})I_p + (\mu+1)\partial_{\mu-1}I_{p+1} + pI_{p-1} = 0, \quad p = 0, 1, \ldots, \mu-2.$$

Applying the differentiation ∂_j to the first and second groups of relations (21)-(22) and making use of cyclic relations (10), it is not difficult to write down the system of differential equations satisfied by the unknown I_p for every $p \geq 0$. In particular, one gets the following relations for $p = 0$:

$$(s_1\partial_0\partial_j + 2s_2\partial_1\partial_j + \cdots + (\mu-1)s_{\mu-1}\partial_{\mu-2}\partial_j + (\mu+1)\partial_{\mu-1}\partial_{j+1} + j\partial_{j-1})I_0 = 0, \quad j = 0, 1, \ldots, \mu-2.$$

Next, applying the differentiation $\partial_0^{\mu-2}$ to the equation with $j = 0$, one gets

$$(s_1\partial_0^{\mu} + 2s_2\partial_0^{\mu-1}\partial_1 + 3s_3\partial_0^{\mu-2}\partial_1^2 + \ldots + (\mu-1)s_{\mu-1}\partial_0^2\partial_1^{\mu-2} + (\mu+1)\partial_1^{\mu})I_0 = 0 \qquad (23)$$

in view of the identity $\partial_1^k(I_0) = \partial_0^{k-1}\partial_k(I_0)$, $1 \leq k \leq \mu$, which follows from the cyclic relations (10). At last, with the help of the second equation of the second group (22) one can eliminate $J_{\mu+1}$ from the first equation of the first group (21)

$$((\mu+1)s_0\partial_0 + \mu s_1\partial_1 + \ldots + 2s_{\mu-1}\partial_{\mu-1} - \rho + \mu)I_0 = 0. \qquad (24)$$

After these preliminary calculations we are able to consider in detail the first example, the case of the two-parameter family associated with an A_μ-singularity, that is, $P_\mu(s,z) = z^{\mu+1} + s_1 z + s_0$. At first remark, that it is possible to restrict both equations (23) and (24) to the subspace $s_2 = \ldots = s_{\mu-1} = 0$; so that the obtained Picard-Fuchs system looks like this:

$$\begin{aligned} &((\mu+1)s_0\partial_0 + \mu s_1\partial_1 - \rho + \mu)I_0 = 0, \\ &(s_1\partial_0^{\mu} + (\mu+1)\partial_1^{\mu})I_0 = 0. \end{aligned} \qquad (25)$$

As a result one gets a slightly corrected system from [9]

$$\begin{array}{l}((\mu+1)\delta_0+\mu\delta_1-\rho+\mu)I_0=0,\\ (\mu^\mu(\mu+1)^{-\mu}x^{-1}(\delta_1)_\mu+(-1)^\mu(\delta_0)_\mu)I_0=0,\end{array} \tag{26}$$

where $\delta_0 = s_0\partial_0$, $\delta_1 = s_1\partial_1$, $(\delta)_\mu = \delta(\delta-1)\dots(\delta-\mu+1)$ is the Pochhammer operator, and $x = (-\mu)^\mu s_1^{\mu+1}/(\mu+1)^{\mu+1}s_0^\mu$.

In the case of the 3-parameter family $P_3(s,z) = z^4+s_2z^2+s_1z+s_0$, the versal deformation of an A_3-singularity, similar manipulations give us the system equivalent to a Picard-Fuchs one (cf. (14) and (15)) for the versal integrals I_0, I_1 and I_2 :

$$\begin{array}{rcl}(s_0\partial_0+s_1\partial_1+s_2\partial_2)I_0+\partial_2I_2+(1-\nu/2)I_0 &=& 0,\\ (s_1\partial_0+2s_2\partial_1)I_0+4\partial_2I_1 &=& 0,\\ (s_1\partial_0+2s_2\partial_1)I_1+4\partial_2I_2+I_0 &=& 0,\\ (4s_0\partial_0+3s_1\partial_1+2s_2\partial_2+2-2\nu)I_1 &=& 0,\\ (4s_0\partial_0+3s_1\partial_1+2s_2\partial_2+1-2\nu)I_2 &=& 0.\end{array}$$

Making use of cyclic relations (10), one gets still one more system from [9] satisfied by the integral I_0 :

$$\begin{array}{rcl}(4s_0\partial_0+3s_1\partial_1+2s_2\partial_2+3-2\nu)I_0 &=& 0,\\ (s_1\partial_0^2+2s_2\partial_1\partial_0+4\partial_2\partial_1)I_0 &=& 0,\\ (s_1\partial_0\partial_1+2s_2\partial_1^2+4\partial_2^2+\partial_0)I_0 &=& 0,\\ (\partial_0\partial_2-\partial_1^2)I_0 &=& 0.\end{array} \tag{27}$$

7 Minimal equations

In this section we derive the minimal equations for systems (25) and (27). In the first case a generalized Gauss-Legendre equation of order μ is obtained; its singular locus coincides with the singular locus of the system. In the second case the singular locus of the minimal equation of the third order is the union of the singular locus of the system and one additional component.

So, in the notations of Section 6 let us consider the Picard-Fuchs system (25) in the case of a two-parameter family. In order to simplify notations below the integral I_0 is denoted by u. The first equation of (25) yields the relation

$$\mu s_1\partial_1(u) = -(\mu+1)s_0\partial_0(u)-(\mu-(\mu+1)\nu/2)u.$$

Let us multiply the second equation of (25) by s_1^μ and eliminate the operator $s_1\partial_1$ with the help of this relation. Making use of the identity $(t\partial_t)_\mu = t^k\partial_t^k$, $\partial_t = \partial/\partial t$, for the Pochhammer operator, one obtains

$$(s_1^{\mu+1}\partial_0^\mu + (\mu+1)s_1^\mu\partial_1^\mu)u = (s_1^{\mu+1}\partial_0^\mu + (\mu+1)\big(D(\alpha,\beta)\big)_\mu)u = 0.$$

where $D(\alpha,\beta) = \alpha s_0\partial_0 + \beta$, $\alpha = -(\mu+1)/\mu$, and $\beta = -\alpha\nu/2 - 1$. Let us determine two groups of constants, a_i, $i = 1,2,\dots,$ and their reciprocals $b_i^{(j)}$, $i \le j$, $i,j = 1,2,\dots,\mu$, by the following relations:

$$(t\partial_t)^k = \sum_{i=1}^k a_i t^i\partial_t^i, \quad (D)_k = D(D-1)\cdots(D-k+1) = \sum_{i=1}^k b_i^{(k)}D^i.$$

For example, the coefficients a_i are determined by the triangular system of linear equations

$$\begin{aligned}
1^k &= a_1,\\
2^k &= \tfrac{2!}{1!}a_2 + \tfrac{2!}{1!}a_1,\\
3^k &= \tfrac{3!}{1!}a_3 + \tfrac{3!}{1!}a_2 + \tfrac{3!}{2!}a_1,\\
&\vdots\\
\ell^k &= \tfrac{\ell!}{1!}a_\ell + \tfrac{\ell!}{1!}a_{\ell-1} + \tfrac{\ell!}{2!}a_{\ell-2} + \ldots + \tfrac{\ell!}{(\ell-1)!}a_1.
\end{aligned}$$

In particular, $a_1 = 1$, $a_2 = 2^{k-1}$, $a_3 = (3^{k-1} - 2^k + 1)/2$, and so on. The proof is an immediate consequence of the two identities: $(t\partial_t)^k t^\ell = \ell! t^\ell$ and $t^k\partial_t^k(t^\ell) = t^\ell\ell!/(l-k)!$, where $k \le \ell$. Now we are able to perform the following transformations

$$\begin{aligned}
&\big(s_1^{\mu+1}\partial_0^\mu + (\mu+1)D(\alpha,\beta)(D(\alpha,\beta)-1)\ldots(D(\alpha,\beta)-\mu+1)\big)u =\\
&(s_1^{\mu+1}\partial_0^\mu + (\mu+1)\textstyle\sum_{i=1}^\mu b_i^{(\mu)}(\alpha s_0\partial_0+\beta)^i)u = (s_1^{\mu+1}\partial_0^\mu +\\
&(\mu+1)(\textstyle\sum_{i=1}^\mu b_i^{(\mu)}\sum_{j=1}^i \binom{i}{j}\alpha^j\beta^{i-j}(s_0\partial_0)^j + \sum_{i=1}^\mu b_i^{(\mu)}\beta^i))u =\\
&(s_1^{\mu+1}\partial_0^\mu + (\mu+1)(\textstyle\sum_{i=1}^\mu b_i^{(\mu)}(\sum_{j=1}^i \binom{i}{j}\alpha^j\beta^{i-j}(\sum_{k=1}^j a_k s_0^k\partial_0^k))+\\
&\textstyle\sum_{i=1}^\mu b_i^{(\mu)}\beta^i))u = 0.
\end{aligned}$$

Computing coefficients of ∂_0^μ, $\partial_0^{\mu-1}$, etc., one gets

$$(\Delta\partial_0^\mu + c_{\mu-1}\Delta'\partial_0^{\mu-1} + \ldots + c_1\Delta^{(\mu-1)}\partial_0 + c_0\Delta^{(\mu)})u = 0, \qquad (28)$$

where $\Delta = (\mu+1)^{\mu+1}s_0^{\mu}+(-1)^{\mu}\mu^{\mu}s_1^{\mu+1}$, $\Delta' = \partial_0(\Delta)$, $\Delta'' = \partial_0^2(\Delta), \ldots,$ and the constants c_i, $i = 0, 1, \ldots, \mu-1$, are determined by the following relations

$$\begin{aligned}
(\mu+1)^{\mu}c_{\mu-1} &= \mu a_{\mu-1}\big(b_{\mu}^{(\mu)}(\alpha^{\mu} + \tbinom{\mu}{\mu-1}\alpha^{\mu-1}\beta) + b_{\mu-1}^{(\mu)}\alpha^{\mu-1}\big),\\
(\mu+1)^{\mu}c_{\mu-2} &= \mu(\mu-1)a_{\mu-2}\big(b_{\mu}^{(\mu)}(\alpha^{\mu} + \tbinom{\mu}{\mu-1}\alpha^{\mu-1}\beta + \\
&\quad \tbinom{\mu}{\mu-2}\alpha^{\mu-2}\beta^2) + \\
&\quad b_{\mu-1}^{(\mu)}(\alpha^{\mu-1} + \tbinom{\mu-1}{\mu-2}\alpha^{\mu-2}\beta) + b_{\mu-2}^{(\mu)}\alpha^{\mu-2}\big),\\
&\vdots\\
(\mu+1)^{\mu}c_0 &= \mu!\sum_{i=1}^{\mu} b_i^{(\mu)}\beta^i.
\end{aligned}$$

Thus, equation (28) of order μ can be considered as a natural generalization of the classical Gauss-Legendre equation of second order (cf. (20) in Section 6). For example, in the case when $\mu = 2$, $\nu = 1$, one gets

$$(\Delta\partial_0^2 + \Delta'\partial_0 + \frac{5}{72}\Delta'')u = 0,$$

where $\Delta = 27s_0^2 + 4s_1^3$.

Let us consider the second example, the versal deformation of an A_3-singularity. With the help of the first equation of system (27) one can express the operator ∂_1 making use of the following identity:

$$s_1\partial_1 = -\tfrac{2}{3}s_2\partial_2 - \tfrac{4}{3}\partial_0 - \tfrac{1}{3},$$

and then eliminate ∂_1^2 and ∂_1 :

$$\begin{aligned}
&((s_1^2 - \tfrac{8}{3}s_0s_2)\partial_0^2 - (\tfrac{4}{3}s_2^2 + \tfrac{2^4}{3}s_0)\partial_0\partial_2 - \tfrac{10}{3}s_2\partial_0 - 4\partial_2 - \tfrac{8}{3}s_2\partial_2^2)u = 0,\\
&(-\tfrac{4}{3}s_0\partial_0^2 + \tfrac{4}{3}s_2\partial_0\partial_2 - \tfrac{2}{3}\partial_0 + 4\partial_2^2)u = 0,\\
&(\tfrac{2^4}{3^2}\partial_0^2 + (\tfrac{2^4}{3^2}s_0s_2 - s_1^2)\partial_0\partial_2 + 4s_0\partial_0 + \tfrac{14}{3^2}s_2\partial_2 + \tfrac{4}{3^2}s_2^2\partial_2^2 + \tfrac{4}{3^2})u = 0.
\end{aligned}$$

At last it is possible to eliminate $4\partial_2^2$ from the first and third equations in virtue of the second one:

$$\begin{aligned}
&((3s_1^2 - \tfrac{2^5}{3})\partial_0^2 - (\tfrac{4}{3}s_2^2 + 2^4s_0)\partial_0\partial_2 - \tfrac{34}{3}s_2\partial_0 - 12\partial_2)u = 0,\\
&((16s_0^2 + \tfrac{4}{3}s_0s_2^2)\partial_0^2 + (36s_0 + \tfrac{2}{3}s_2^2)\partial_0 + \\
&\qquad (16s_0s_2 - 9s_1^2 - \tfrac{4}{3}s_2^3)\partial_0\partial_2 + 14s_2\partial_2 + 4)u = 0,
\end{aligned}$$

and to solve this system for $\partial_0\partial_2$ and ∂_2. As a result one gets:

$$\begin{aligned}
P\partial_2 &= \Delta\partial_0^2 + A_0\partial_0 + A,\\
P\partial_0\partial_2 &= -2B_{00}\partial_0^2 - 2B_0\partial_0 - 2B,
\end{aligned}$$

where Δ is the discriminant (16), and

$$\begin{aligned}
P &= -4(\tfrac{26}{3}s_2^3 + 2^3 s_2 s_0 + 3^2 s_1^2),\\
A_0 &= -\tfrac{2^4\cdot 23}{3}s_0 s_2^2 + 102 s_1^2 s_2 + 2^4 s_2^4 + 2^6 3^2 s_0^2,\\
A &= \tfrac{2^4}{3}s_2^2 + 2^6 s_0,\\
B_{00} &= 21 s_1^2 s_2 - \tfrac{2^3 5^2}{3}s_0 s_2^2 + 2^5 3 s_0^2,\\
B_0 &= -\tfrac{2\cdot 113}{3}s_2^2 + 2^3 3^3 s_0,\\
B &= 24.
\end{aligned}$$

Further, $\partial_0(P\partial_0) = P\partial_0\partial_2 + (\partial_0 P)\partial_2 = \partial_0(\Delta\partial_0^2 + A_0\partial_0 + A)$, that is,

$$(\Delta\partial_0^3 + (\Delta' + A_0 + 2B_{00})\partial_0^2 + (A_0' + A + 2B_0)\partial_0 + A' + 2B + 2^5 s_2\partial_2)u = 0,$$

where the partial derivative with respect to s_0 is denoted by prime. Multiplying by P and eliminating $P\partial_2$, one gets

$$\begin{aligned}
&(P\Delta\partial_0^3 + (\Delta' P + A_0 P + 2B_{00}P + 2^5 s_2\Delta)\partial_0^2 +\\
&(A_0' P + AP + 2B_0 P + 2^5 s_2 A_0)\partial_0 + A' P + 2BP + 2^5 s_2 A)u = 0.
\end{aligned}$$

As a result, the minimal equation takes the form:

$$(P\Delta\partial_0^3 + \gamma_2\partial_0^2 + \gamma_1\partial_0 + \gamma_0)u = 0,$$

where Δ is given by the expression (16). The singular locus of this equation is the union of the discriminant set and zeros of the polynomial P. However, as was also remarked before, the singular locus of the system in question coincides with the discriminant set. Moreover, Proposition 4 implies that this system can be reduced to an equation of sixth order with singularities along the discriminant only. Its highest derivative coefficient is equal to Δ^6 up to a constant factor.

8 Fundamental solutions

Now we are going to determine solutions of the above holonomic systems for the integral $u = I_0$ in an explicit form. First let us consider the case of 2-parameter versal family of an A_k-singularity. It is possible to rewrite the system (25) or, equivalently, (26) in the following form

$$(k+1)s_0\partial_0 u + k s_1\partial_1 u + (k-\rho)u = 0, \tag{29}$$

$$(k+1)\partial_1^k u + s_1\partial_0^k u = 0. \tag{30}$$

As was remarked in [9], the fundamental solution of system (26) can be expressed via a generalized hypergeometric function. Now we describe a detailed proof of this result and correct some insignificant errors and misprints. Thus, one has to determine the following series

$$u = \sum_{\alpha,\beta} u_{\alpha\beta} s_0^\alpha s_1^\beta.$$

The range of values of the indices α and β will be defined below. Substituting the series in (29), (30), one obtains

$$(k+1)\alpha + k\beta + (k-\rho) = 0, \tag{31}$$

$$\sum (k+1)(-\beta)_k u_{\alpha\beta} s_0^\alpha s_1^{\beta-k} + (-\alpha)_k u_{\alpha\beta} s_0^{\alpha-k} s_1^\beta = 0, \tag{32}$$

where $(\delta)_k = \delta(\delta+1)\cdots(\delta+k-1)$ is the Pochhammer symbol. The first relation (31) implies

$$\alpha = \alpha(\beta) = \alpha_0 - \frac{k}{k+1}\beta, \qquad \text{where} \quad \alpha_0 = \frac{\nu}{2} - \frac{k}{k+1}.$$

That is, the index α is determined by the index β. Let us redefine the unknowns $u_\beta = u_{\alpha\beta}$, and set $t = s_0^{-k/(k+1)} s_1$. Then the second relation (32) yields

$$\sum_\beta (k+1)(-\beta)_k u_\beta t^{\beta-k-1} + (-\alpha)_k u_\beta t^\beta = 0.$$

As a result, equating all the coefficients to zero, one obtains

$$(k+1)(-\beta-k-1)_k u_{\beta+k+1} + (-\alpha)_k u_\beta = 0.$$

Making use of this relation one can compute k linearly independent solution-series of the fundamental solution of the above holonomic system, which are determined by their first terms $u_m = 1, m = 0, \ldots, k-1$, in the following form

$$u_{m+n(k+1)} = \frac{(-1)^n \prod_{j=1}^{n} (-\alpha(\beta_j))_k}{(k+1)^n \prod_{j=1}^{n} (-\beta_j)_k},$$

where $\beta_j = m + j(k+1)$. It is possible to express both products via the Pochhammer symbols appearing in the generalized hypergeometric

series:

$$\prod_{j=1}^{n}(-\alpha(\beta_j))_k = k^n \prod_{i=0}^{k-1}(a_i)_n, \qquad \text{where} \quad a_i = \frac{m+i-\alpha_0}{k+1}+1,$$

$$\prod_{j=1}^{n}(-\beta_j)_k = (-1)^n(k+1)^n \prod_{i=0, i\neq m}^{k-1}(b_i)_n n!, \qquad b_i = \frac{m-i}{k+1}+1.$$

Thus for every $m = 0, 1, \ldots, k-1$, the solution u is the product of the monomial and the generalized hypergeometric series in x :

$$u = s_0^{\alpha(m)} s_1^m \sum_{n=0}^{\infty} \frac{\prod_{i=0}^{k-1}(a_i)_n}{\prod_{i=0,\, i\neq m}^{k-1}(b_i)_n}\frac{x^n}{n!}, \quad x = \frac{k}{(k+1)^2}t^{k+1}. \tag{33}$$

Before the analysis of the second example let us now recall a generalization of Horn's definition due to [4]. A power series

$$\sum_{n_1,\ldots,n_k} a_{n_1,\ldots,n_k} x_1^{n_1}\cdots x_k^{n_k},$$

is called an k-tuple hypergeometric series if the quotients

$$\varphi_j(n_1,\ldots,n_k) = \frac{a_{n_1,\ldots,n_j+1,\ldots,n_k}}{a_{n_1,\ldots,n_j,\ldots,n_k}}, \quad j \in \{1,\ldots,k\},$$

are rational functions of the variables $n_1, \ldots, n_k$. It will be also called a k-dimensional hypergeometric series of Horn's type. Hence, the series in formula (33) can be considered as a 1-dimensional hypergeometric series of Horn's type. The aim of the calculations below is to establish a similar result in the case of the versal deformation of an A_3-singularity. It is not difficult to verify that the system (27) can be expressed in the following form

$$(4\delta_0 + 3\delta_1 + 2\delta_2 - (2\nu - 3))u = 0, \tag{34}$$

$$(4x^{-1}y^{-2}\delta_1\delta_2 + 2x^{-1}\delta_0\delta_1 + y^{-1}\delta_0(\delta_0 - 1))u = 0, \tag{35}$$

$$(4y^{-3}\delta_2(\delta_2 - 1) + 2x^{-1}\delta_1(\delta_1 - 1) + y^{-1}\delta_0(\delta_1 + 1))u = 0, \tag{36}$$

$$(y^{-1}\delta_0\delta_2 - x^{-1}\delta_1(\delta_1 - 1))u = 0, \tag{37}$$

where $\delta_i = s_i\partial/\partial s_i$, $i = 0, 1, 2$, are the Pochhammer operators of first order (see [9]). Let us take the first integrals $x = s_0^{-3/2}s_1^2$ and

$y = s_0^{-1/2} s_2$ of equation (34) as new variables and suppose that the solutions can be presented as fractional power series

$$u = s_0^{\nu/2-3/4} \sum_{n,m} u_{nm} x^n y^m, \tag{38}$$

where all the indices n and m are non-negative rational numbers. Making use of the following rules it is easy to write down the system of finite difference equations for the coefficients u_{nm}. By definition, the action of the Pochhammer operators δ_0, δ_1 and δ_2 on the monomial $s_0^{\frac{1}{2}\nu-\frac{3}{4}} x^n y^m$ is equivalent to multiplication by $p = p_{nm} = -\frac{3}{2}n - \frac{1}{2}m + \frac{1}{2}\nu - \frac{3}{4}$, by $2n$ and by m, respectively. After multiplication by x^{-1} (respectively by y^{-1}) the index n (respectively m) of coefficients increases by one. As a result the system of differential equations (35)-(37) is reduced to the following system of finite difference equations:

$$8(n+1)(m+2)u_{n+1\,m+2} + 4(n+1)(p-\tfrac{3}{4})u_{n+1\,m} + (p-\tfrac{1}{2})(p-\tfrac{3}{2})u_{n\,m+1} = 0, \tag{39}$$

$$4(m+2)(m+3)u_{n\,m+3} + 4(n+1)(2n+1)u_{n+1\,m} + (2n+1)(p-\tfrac{1}{2})u_{n\,m+1} = 0, \tag{40}$$

$$(m+1)(p-\tfrac{1}{2})u_{n\,m+1} - 2(n+1)(2n+1)u_{n+1\,m} = 0. \tag{41}$$

Since the pattern of this system of difference equations is a parallelogram it can be easy solved. In addition the latter equation simplifies the task since it connects two adjacent vertices of the parallelogram. Let us reindex the equation (41)

$$(m+3)(p-\tfrac{3}{2})u_{n\,m+3} - 2(n+1)(2n+1)u_{n+1\,m+2} = 0,$$

and multiply equation (39) by $(2n+1)$. Then $u_{n+1\,m}$ (respectively $u_{n+1\,m+2}$) is eliminated with the help of equation (41) (respectively of the reindexed equation (41)) :

$$4(m+2)(m+3)(p-\tfrac{3}{2})u_{n\,m+3} + 2(m+1)(p-\tfrac{3}{4})(p-\tfrac{1}{2})u_{n\,m+1} + (2n+1)(p-\tfrac{1}{2})(p-\tfrac{3}{2})u_{n\,m+1} = 0. \tag{42}$$

Analogously $u_{n+1\,m}$ is eliminated from equation (40) :

$$4(m+2)(m+3)u_{n\,m+3} + 2(m+1)(p-\tfrac{1}{2})u_{n\,m+1} + (2n+1)(p-\tfrac{1}{2})u_{n\,m+1} = 0,$$

that is,

$$4(m+2)(m+3)u_{n\ m+3} + (2m+2n+3)(p-\tfrac{1}{2})u_{n\ m+1} = 0. \quad (43)$$

Suppose $(p-\frac{3}{2}) \neq 0$. It is evident that equation (43) multiplied by $(p-\frac{3}{2})$ gives equation (42). Hence equation (39) can be omitted. Reindexing equation (41), one obtains

$$u_{nm} = \frac{(m+1)(p+1)}{4n(n-\frac{1}{2})}u_{n-1\ m+1} = \frac{(m+1)_n(p+1)_n}{4^n n!(\frac{1}{2})_n}u_{0\ n+m}. \quad (44)$$

In a similar way equation (43) yields

$$u_{0\ 2k} = \frac{(k-\frac{3}{4})(k-\frac{\nu}{2}-\frac{1}{4})}{4k(k-\frac{1}{2})}u_{0\ 2(k-1)} = \frac{(\frac{1}{4})_k(-\frac{\nu}{2}+\frac{3}{4})_k}{4^k k!(\frac{1}{2})_k}u_{0\ 0}, \quad (45)$$

$$u_{0\ 2k+1} = \frac{(k-\frac{1}{4})(k-\frac{\nu}{2}+\frac{1}{4})}{4k(k+\frac{1}{2})}u_{0\ 2k-1} = \frac{(\frac{3}{4})_k(-\frac{\nu}{2}+\frac{5}{4})_k}{4^k k!(\frac{3}{2})_k}u_{0\ 1}. \quad (46)$$

Let us combine subsequently equation (44) with (45) and (46). As a result one obtains the following expressions for the non-zero terms of the fractional power series representing two components of the fundamental solution:

$$u_{n\ 2k-n} = \frac{(-2k)_n(k-\frac{\nu}{2}+\frac{3}{4})_n(\frac{1}{4})_k(-\frac{\nu}{2}+\frac{3}{4})_k}{4^{n+k}n!k!(\frac{1}{2})_k(\frac{1}{2})_n}u_{0\ 0},$$

or, equivalently,

$$u_{n\ 2k-n} = \frac{(-2k)_n(\frac{1}{4})_k(-\frac{\nu}{2}+\frac{3}{4})_{n+k}}{4^{n+k}n!k!(\frac{1}{2})_k(\frac{1}{2})_n}u_{0\ 0}, \quad (47)$$

where $0 \leq n \leq 2k,\ k \geq 0$, and

$$u_{n\ 2k-n+1} = \frac{(-2k-1)_n(k-\frac{\nu}{2}+\frac{5}{4})_n(\frac{3}{4})_k(-\frac{\nu}{2}+\frac{5}{4})_k}{4^{n+k}n!k!(\frac{3}{2})_k(\frac{1}{2})_n}u_{0\ 1},$$

or, equivalently,

$$u_{n\ 2k-n+1} = \frac{(-2k-1)_n(\frac{3}{4})_k(-\frac{\nu}{2}+\frac{5}{4})_{n+k}}{4^{n+k}n!k!(\frac{3}{2})_k(\frac{1}{2})_n}u_{0\ 1}, \quad (48)$$

where $0 \leq n \leq 2k+1,\ k \geq 0$. The remaining third component of the fundamental solution depends on the arbitrary constant $u_{\frac{1}{2}\ 0}$. To be more precise, equation (44) implies

$$u_{n+\frac{1}{2}\ m} = \frac{(m+1)(p+\frac{1}{4})}{4n(n+\frac{1}{2})}u_{n-\frac{1}{2}\ m+1} = \frac{(m+1)_n(p+\frac{1}{4})_n}{4^n n!(\frac{3}{2})_n}u_{\frac{1}{2}\ n+m}, \quad (49)$$

while equation (43) yields

$$u_{\frac{1}{2}\,2k} = \frac{(k-\frac{\nu}{2}+\frac{1}{2})}{4k} u_{\frac{1}{2}\,2(k-1)} = \frac{(-\frac{\nu}{2}+\frac{3}{2})_k}{4^k k!} u_{\frac{1}{2}\,0}. \tag{50}$$

Combining (49) and (50), one gets the following expressions for non-zero terms of the fractional power series representing the third component of the fundamental solution

$$u_{n+\frac{1}{2}\,2k-n} = \frac{(2k-n+1)_n(-n-k+\frac{\nu}{2}-\frac{1}{2})_n(-\frac{\nu}{2}+\frac{3}{2})_k}{4^{n+k}n!k!(\frac{3}{2})_n} u_{\frac{1}{2}\,0},$$

or, equivalently,

$$\begin{aligned} u_{n+\frac{1}{2}\,2k-n} &= \frac{(-2k)_n(k-\frac{\nu}{2}+\frac{3}{2})_n(-\frac{\nu}{2}+\frac{3}{2})_k}{4^{n+k}n!k!(\frac{3}{2})_n} u_{\frac{1}{2}\,0} \\ &= \frac{(-2k)_n(-\frac{\nu}{2}+\frac{3}{2})_{n+k}}{4^{n+k}n!k!(\frac{3}{2})_n} u_{\frac{1}{2}\,0}, \end{aligned} \tag{51}$$

where $0 \le n \le 2k$. It remains to analyze the case when $(p-\frac{3}{2})=0$, that is, $\frac{1}{2}\nu-\frac{3}{4}-\frac{3}{2}n-\frac{1}{2}m=0$. Evidently, in such a case the indices n and m are not integers. On the other hand, when $p_{n+\frac{1}{2}\,2k-n}=\frac{3}{2}$, that is, $n+k=\frac{1}{2}\nu-3$, then (39) implies $u_{n+\frac{3}{2}\,2k-n+2}=0$. However, the third component of the fundamental solution has zero coefficients with such indices as was computed before. In addition, it is also possible to verify that the system (39)-(41) has no nontrivial solutions with zero terms $u_{0\,0}$, $u_{0\,1}$, $u_{\frac{1}{2}\,0}$, without any restrictions on indices n, m. As a result one obtains all the non-zero terms of the three linearly independent components (47), (48) and (51) of the fundamental solution of the maximally overdetermined system of differential equations (39)-(41) at points of the parameter domain $s_1=s_2=0$, $s_0\neq 0$, contained in the complement of D, the discriminant (16), which is the singular locus of the system in question. It is clear that two components (47) and (48) are 2-tuple hypergeometric series of Horn's type while the third one (51) can be considered as such a series depending on two variables $(\sqrt{x}, y)$.

In the note [9] the system (34) - (37) is considered; the author writes down three power series which are two-dimensional hypergeometric functions of Horn's type. One of them is a series centered at points of the domain $s_1=s_2=0$, $s_0\neq 0$, while two other series are centered at points of the line $s_0=s_1=0$, $s_2\neq 0$, contained in the

singular locus of the system. Since the discriminant D is the singular locus of the system, these two series are multi-valued analytic functions branching at points of D. It was also stated there that these three series represent all linearly independent components of the fundamental solution of the system outside the discriminant. However, it still remains unclear whether they are linearly independent in their common domain of analytic continuation. In contrast with [9], three series (47), (48) and (51) are linearly independent by construction; they are centered at points outside of the discriminant D.

For completeness it should be also remarked that one knows explicit expressions for two different kinds for particular solutions of the Picard-Fuchs systems associated with A_μ-singularities, $\mu \geq 2$, (see [11], [6], [2]); in the case of an A_3-singularity some interesting considerations in connection with this problem are described in [15].

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Basic boundary value problems in complex analysis

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Abstract

In order to develop a theory of boundary value problems for complex partial differential equations of arbitrary order at first the three basic boundary value problems for analytic functions and more generally for the inhomogeneous Cauchy-Riemann equation are investigated. They are the Schwarz, the Dirichlet and the Neumann boundary value problem. In order to give explicit expressions as well for the solutions as for the solvability conditions the particular case of the unit disc is studied. Any domain admitting a complex Green function for the Laplacian would allow analogue considerations. The Cauchy-Riemann equation is the model equation for the generalized Beltrami equation. Using the complex potential method the latter equation can be transformed into a singular integral equation the solution of which will provide a solution to the differential equation satisfying the prescribed boundary conditions.

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1 Introduction

An analytic function is determined up to an additive imaginary constant through the boundary values of its real part. The solution to this Schwarz problem is provided by the Schwarz operator expressed through the complex harmonic Green function, see e.g. [1]. Hence the Dirichlet problem seeking an analytic function for given boundary values is overdetermined and thus only conditionally solvable. This is also true for the Neumann boundary value problem prescribing the normal derivative of the function considered on the boundary of the domain. In the case of analytic functions or more generally of solutions to the inhomogeneous Cauchy-Riemann equation or the generalized Beltrami equation a solution of a first order partial differential equation is looked for with prescribed first order derivatives on the boundary.

In order to determine and solve proper boundary value problems for higher order complex partial differential equations here a thorough study of these three boundary value problems are presented. In some further papers second order model equations i.e. the inhomogeneous Laplace, that is the Poisson, and the inhomgeneous Bitsadze equation are treated. Later on then higher order model equations among them the inhomogeneous polyanalytic and the inhomogeneous polyharmonic equation will be investigated.

These results were at first presented at some lectures at the University Simón Bolivar in Caracas in May 2004. The author is grateful to the Department of Mathematics of this university for the support and hospitality, in particular to Prof. Dr. C.J. Vanegas.

2 Basic integral representations

The Cauchy-Pompeiu integral formula is fundamental for solving boundary value problems for complex first order partial differential equations. It is a consequence of the complex Gauss theorem.

Gauss Theorem *(complex forms) Let* $w \in C^1(D;\mathbb{C}) \cap C(\bar{D};\mathbb{C})$ *for a regular domain* D *of the complex plane* $\mathbb{C}$ *then*

$$\int\limits_D w_{\bar{z}}(z)dxdy = \frac{1}{2i}\int\limits_{\partial D} w(z)dz \tag{1}$$

and

$$\int_D w_z(z)dxdy = -\frac{1}{2i}\int_{\partial D} w(z)d\bar{z}\ . \qquad (1')$$

Cauchy-Pompeiu representations *Let $D \subset \mathbb{C}$ be a regular domain and $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$. Then using $\zeta = \xi + i\eta$ for $z \in D$*

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z} \qquad (2)$$

and

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\bar{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}} \qquad (2')$$

hold.

Definition *For $f \in L_1(D;\mathbb{C})$ the integral operator*

$$Tf(z) = -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z}\ ,\ z \in \mathbb{C}\ ,$$

is called Pompeiu operator.

The Pompeiu operator, see [8], is investigated in detail in connection with the theory of generalized analytic functions in Vekua's book [10], see also [1]. Its differentiability properties are important here in the sequal. For generalizations, see e.g. [2], for applications [3, 4].

For $f \in L_1(D;\mathbb{C})$ the Pompeiu integral is differentiable in distributional sense with respect to $\bar{z}$ satisfying in D

$$\partial_{\bar{z}} Tf = f.$$

Moreover, Tf is analytic outside $\bar{D}$ and vanishing of infinity for bounded D. If $f \in L_p(D;\mathbb{C}), 1 < p$, then Tf is differentiable in distributional sense almost everywhere in D and

$$\partial_z Tf(z) = \Pi f(z) = -\frac{1}{\pi}\int_D f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}\ ,$$

where the integral on the right-hand side is understood as a Cauchy principal value integral

$$\int_D f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2} = \lim_{\varepsilon \to 0}\int_{D\setminus\overline{K_\varepsilon(z)}} f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}.$$

Here $K_\varepsilon(z) = \{|\zeta - z| < \varepsilon\}$. This is a deep result of Calderon-Zygmund [5].

Theorem 1 *Any* $w \in C^1(\mathbb{D};\mathbb{C}) \cap C(\overline{\mathbb{D}};\mathbb{C}), \mathbb{D} = \{|z| < 1\}$ *the unit disc, is representable as*

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \text{Re } w(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} + \frac{1}{2\pi}\int\limits_{|\zeta|=1} \text{Im } w(\zeta)\frac{d\zeta}{\zeta} \\ &\quad -\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big(\frac{w_{\bar\zeta}(\zeta)}{\zeta-z} + \frac{zw_{\bar\zeta}(\zeta)}{1-z\bar\zeta}\Big)d\xi d\eta,\ |z|<1\ . \end{aligned} \tag{3}$$

Corollary 1 *Any* $w \in C^1(\mathbb{D};\mathbb{C}) \cap C(\overline{\mathbb{D}};\mathbb{C})$ *can be represented as*

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \text{Re } w(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} + i\ \text{Im } w(0) \\ &\quad -\frac{1}{2\pi}\int\limits_{|\zeta|<1}\Big(\frac{w_{\bar\zeta}(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{w_{\bar\zeta}(\zeta)}}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\Big)d\xi d\eta. \end{aligned} \tag{4}$$

Proof For fixed $z, |z|<1$, the complex Gauss theorem applied to $\mathbb{D}$ shows

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} w(\zeta)\frac{\bar z d\zeta}{1-\bar z\zeta} - \frac{1}{\pi}\int\limits_{|\zeta|<1} w_{\bar\zeta}(\zeta)\frac{\bar z}{1-\bar z\zeta}\ d\xi d\eta = 0\ .$$

Taking the complex conjugate and adding this to (2) in the case $D = \mathbb{D}$ gives for $|z|<1$

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\Big(\frac{\zeta w(\zeta)}{\zeta-z} + \frac{z\overline{w(\zeta)}}{\zeta-z}\Big)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{|\zeta|<1}\Big(\frac{w_{\bar\zeta}(\zeta)}{\zeta-z} + \frac{z\overline{w_{\bar\zeta}}(\zeta)}{1-z\bar\zeta}\Big)d\xi d\eta\ ,$$

where $\bar\zeta d\zeta = -\zeta d\bar\zeta$ for $|\ \zeta\ | = 1$ is used. This is (3). Subtracting i Im $w(0)$ from (3) proves (4).

Remark For analytic functions (4) is the Schwarz-Poisson formula

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \text{Re } w(\zeta)\Big(\frac{2\zeta}{\zeta-z} - 1\Big)\frac{d\zeta}{\zeta} + i\ \text{Im } w(0)\ . \tag{4'}$$

The kernel

$$\frac{\zeta + z}{\zeta - z} = \frac{2\zeta}{\zeta - z} - 1$$

is called the Schwarz kernel. Its real part

$$\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\overline{\zeta - z}} - 1 = \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2}$$

is the Poisson kernel. The Schwarz operator

$$S\varphi(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \varphi(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta}$$

for $\varphi \in C(\partial\mathbb{D}; \mathbb{R})$ is known to provide an analytic function in $\mathbb{D}$ satisfying

$$\text{Re } S\varphi = \varphi \text{ on } \partial\mathbb{D},$$

see [9], in the sense

$$\lim_{z \to \zeta} S\varphi(z) = \varphi(\zeta) \ , \ \zeta \in \partial\mathbb{D} \ ,$$

for z in $\mathbb{D}$ tending to ζ. Poisson has proved the respective representation in particular for harmonic functions, i.e. to solutions for the Laplace equation

$$\Delta u = \partial_x^2 u + \partial_y^2 u = 0$$

in $\mathbb{D}$, see e.g. the references in [8]. Re w for analytic w is harmonic. The Schwarz operator can be defined for other simply and even multiconnected domains, see e.g. [1].

Formula (4) is called the Cauchy-Schwarz-Poisson-Pompeiu formula. Rewriting it according to

$$w_{\bar{z}} = f \text{ in } \mathbb{D} \ , \ \text{Re } w = \varphi \text{ on } \partial\mathbb{D} \ , \ \text{Im } w(0) = c \ ,$$

then

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \varphi(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + ci$$

$$- \frac{1}{2\pi} \int\limits_{|\zeta|<1} \Big(\frac{f(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} \Big) d\xi d\eta \qquad (4'')$$

is expressed by the given data. Applying the result of Schwarz it is easily seen that taking the real part on the right-hand side and letting

z tend to a boundary point ζ this tends to $\varphi(\zeta)$. Differentiating with respect to $\bar{z}$ as every term on the right-hand side is analytic besides the T-operator applied to f this gives $f(z)$. Also for $z = 0$ besides ic all other terms on the right-hand side are real. Hence, $(4'')$ is a solution to the so-called Dirichlet problem

$$w_{\bar{z}} = f \text{ in } \mathbb{D},\ \text{Re } w = \varphi \text{ on } \partial\mathbb{D}\ ,\ \text{Im } w(0) = c\ .$$

This shows how integral representation formulas serve to solve boundary value problems. The method is not restricted to the unit disc but in this case the solutions to the problems are given in an explicit way.

3 Basic boundary value problems

As was pointed out in connection with the Schwarz-Poisson formula in the case of the unit disc boundary value problems can be solved explicitly. For this reason this particular domain is considered. This will give necessary information about the nature of the problems considered. The simplest and therefore fundamental cases occur with respect to analytic functions.

Schwarz boundary value problem *Find an analytic function w in the unit disc, i.e. a solution to $w_{\bar{z}} = 0$ in $\mathbb{D}$, satisfying*

$$Re\ w = \gamma\ on\ \partial\mathbb{D},\ \ Im\ w(0) = c$$

for $\gamma \in C(\partial\mathbb{D}; \mathbb{R}), c \in \mathbb{R}$ given.

Theorem 2 *This Schwarz problem is uniquely solvable. The solution is given by the Schwarz formula*

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + ic\ . \tag{5}$$

The proof follows from the Schwarz-Poisson formula $(4')$ together with a detailed study of the boundary behaviour, see [9].

Dirichlet boundary value problem *Find an analytic function w in the unit disc, i.e. a solution to $w_{\bar{z}} = 0$ in $\mathbb{D}$, satisfying for given $\gamma \in C(\partial\mathbb{D}; \mathbb{C})$*

$$w = \gamma\ on\ \partial\mathbb{D}\ .$$

Theorem 3 *This Dirichlet problem is solvable if and only if for* $|z|<1$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta}=0\ . \tag{6}$$

The solution is then uniquely given by the Cauchy integral

$$w(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{\zeta-z}\ . \tag{7}$$

Remark This result is a consequence of the Plemelj-Sokhotzki formula, see e.g. [1, 6, 7]. The Cauchy integral (7) obviously provides an analytic function in $\mathbb{D}$ and one in $\hat{\mathbb{C}}\setminus\bar{\mathbb{D}},\hat{\mathbb{C}}$ the Riemann sphere. The Plemelj-Sokhotzki formula states that in case of Hölder continuous γ for $|\zeta|=1$

$$\lim_{z\to\zeta,|z|<1}w(z)-\lim_{z\to\zeta,1<|z|}w(z)=\gamma(\zeta)\ .$$

In order that for any $|\zeta|=1$

$$\lim_{z\to\zeta,|z|<1}w(z)=\gamma(\zeta)$$

the condition

$$\lim_{z\to\zeta,1<|z|}w(z)=0$$

is necessary and sufficient. However, the Plemelj-Sokhotzki formula in its classical formulation holds if γ is Hölder continuous. Nevertheless, for the unit disc Hölder continuity is not needed, see [7].

Proof 1. (6) is shown to be necessary. Let w be a solution to the Dirichlet problem. Then w is analytic in $\mathbb{D}$ having continuous boundary values

$$\lim_{z\to\zeta}w(z)=\gamma(\zeta) \tag{8}$$

for all $|\zeta|=1$.
Consider for $1<|z|$ the function

$$w\Big(\frac{1}{\overline{z}}\Big)=-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta}=-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\overline{z}}{\overline{\zeta-z}}\frac{d\zeta}{\zeta}\ .$$

As with $z, 1 < |z|$, tending to $\zeta, |\zeta| = 1, 1/\bar{z}$ tends to ζ too, $\lim_{z\to\zeta} w(1/\bar{z})$ exists, i.e. $\lim_{z\to\zeta} w(z)$ exists for $1 < |z|$. From

$$w(z) - w\Big(\frac{1}{\bar{z}}\Big) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta)\Big(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\overline{\zeta - z}} - 1\Big)\frac{d\zeta}{\zeta}$$

and the properties of the Poisson kernel for $|\,\zeta\,| = 1$

$$\lim_{z\to\zeta,|z|<1} w(\zeta) - \lim_{z\to\zeta,1<|z|} w(z) = \gamma(\zeta) \tag{9}$$

follows. Comparison with (8) shows $\lim_{z\to\zeta} w(z) = 0$ for $1 <|\, z\,|$. As $w(\infty) = 0$ then the maximum principle for analytic functions tells that $w(z) \equiv 0$ in $1 <|\, z\,|$. This is condition (6).
2. The sufficiency of (6) follows at once from adding (6) to (7) leading to

$$\begin{aligned} w(z) &= \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta)\Big(\frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\overline{\zeta - z}}\Big)\frac{d\zeta}{\zeta} \\ &= \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta)\Big(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\overline{\zeta - z}} - 1\Big)\frac{d\zeta}{\zeta}\ . \end{aligned}$$

Thus for $|\,\zeta\,| = 1$

$$\lim_{z\to\zeta,|z|<1} w(z) = \gamma(\zeta)$$

follows again from the properties of the Poisson kernel.

The third basic boundary value problem is based on the outward normal derivative at the boundary of a regular domain. This directional derivative on a circle $|\,z - a\,| = r$ is in the direction of the radius vector, i.e. the outward normal vector is $\nu = (z-a)/r$, and the normal derivative in this direction ν given by

$$\partial_\nu = \partial_r = \frac{z-a}{r}\ \partial_z + \frac{\overline{z-a}}{r}\ \partial_{\bar{z}}\ .$$

In particular for the unit disc $\mathbb{D}$

$$\partial_r = z\partial_z + \bar{z}\partial_{\bar{z}}\ .$$

Neumann boundary value problem *Find an analytic function w in the unit disc, i.e. a solution to $w_{\bar{z}} = 0$ in $\mathbb{D}$, satisfying for some $\gamma \in C(\partial\mathbb{D};\mathbb{C})$ and $c \in \mathbb{C}$*

$$\partial_\nu w = \gamma \text{ on } \partial\mathbb{D}\ ,\ w(0) = c\ .$$

Theorem 4 *This Neumann problem is solvable if and only if*

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{(1-\bar{z}\zeta)\zeta}=0 \tag{10}$$

is satisfied for $|z|<1$. *The solution then is*

$$w(z)=c-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\log(1-z\bar{\zeta})\frac{d\zeta}{\zeta}\ . \tag{11}$$

Proof The boundary condition reduces to the Dirichlet condition

$$zw'(z)=\gamma(z)\ \text{for}\ \ |z|=1$$

because of the analyticity of w. Hence from the preceding result

$$zw'(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{\zeta-z}$$

if and only if for $|z|<1$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\bar{z}d\zeta}{1-\bar{z}\zeta}=0\ . \tag{12}$$

But as $zw'(z)$ vanished at the origin this imposes the additional condition

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{\zeta}=0 \tag{13}$$

on γ. Then

$$w'(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{(\zeta-z)\zeta}\ .$$

Integrating shows

$$w(z)=c-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\log\frac{\zeta-z}{\zeta}\frac{d\zeta}{\zeta}$$

which is (11). Adding (12) and (13) leads to

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{1}{1-\bar{z}\zeta}\frac{d\zeta}{\zeta}&=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\bar{\zeta}}{\overline{\zeta-z}}\frac{d\zeta}{\zeta}\\&=-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\bar{\zeta}}{\overline{\zeta-z}}=0\ ,\end{aligned}$$

i.e. to (10). By integration this gives

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\log(1-\bar{z}\zeta)d\bar{\zeta}=0\ .$$

Next these boundary value problems will be studied for the inhomogeneous Cauchy-Riemann equation. Using the T-operator the problems will be reduced to the ones for analytic functions. Here in the case of the Neumann problem it will make a difference if the normal derivative on the boundary or only the effect of $z\partial_z$ on the function is prescribed.

Theorem 5 *The Schwarz problem for the inhomogeneous Cauchy-Riemann equation in the unit disc*

$$w_{\bar{z}}=f\ in\ \mathbb{D}\ ,\ Re\ w=\gamma\ on\ \partial\mathbb{D}\ ,\ Im\ w(0)=c$$

for $f\in L_1(\mathbb{D};\mathbb{C}),\gamma\in C(\partial\mathbb{D};\mathbb{R}),c\in\mathbb{R}$ *is uniquely solvable by the Cauchy-Schwarz-Pompeiu formula*

$$\begin{aligned}w(z) &= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}+ic\\ &\quad -\frac{1}{2\pi}\int\limits_{|\zeta|<1}\Big[\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{f(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\Big]d\xi d\eta\ .\end{aligned}\qquad(14)$$

This representation (14) follows just from (4) assuming that the solution w exists. But (14) can easily be verified to be a solution. That this solution is unique follows from Theorem 2.

Theorem 6 *The Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in the unit disc*

$$w_{\bar{z}}=f\ in\ \mathbb{D}\ ,\ w=\gamma\ on\ \partial\mathbb{D}$$

for $f\in L_1(\mathbb{D};\mathbb{C})$ *and* $\gamma\in C(\partial\mathbb{D};\mathbb{C})$ *is solvable if and only if for* $|z|<1$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\bar{z}d\zeta}{1-\bar{z}\zeta}=\frac{1}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{\bar{z}d\xi d\eta}{1-\bar{z}\zeta}\ .\qquad(15)$$

The solution then is uniquely given by

$$w(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{\zeta-z}-\frac{1}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{d\xi d\eta}{\zeta-z}\ .\qquad(16)$$

Representation (16) follows from (2) if the problem is solvable. The unique solvability is a consequence of Theorem 3. That (16) actually is a solution under (15) follows by observing the properties of the T-operator on one hand and from

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\Big(\frac{\zeta}{\zeta-z}+\frac{\bar{\zeta}}{\overline{\zeta-z}}-1\Big)\frac{d\zeta}{\zeta} \\ &\quad -\frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\Big(\frac{1}{\zeta-z}+\frac{\bar{z}}{1-\bar{z}\zeta}\Big)d\xi d\eta = \gamma(z) \end{aligned}$$

for $|\,z\,|=1$ on the other.

That (15) is also necessary follows from Theorem 3. Applying condition (6) to the boundary value of the analytic function $w-Tf$ in $\mathbb{D}$, i.e. to $\gamma - Tf$ on $\partial\mathbb{D}$ gives (15) because of

$$\begin{aligned} &\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta})\frac{d\tilde{\xi}d\tilde{\eta}}{\tilde{\zeta}-\zeta}\frac{\bar{z}d\zeta}{1-\bar{z}\zeta} \\ &= -\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta})\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\bar{z}}{1-\bar{z}\zeta}\frac{d\zeta}{\zeta-\tilde{\zeta}}\,d\tilde{\xi}d\tilde{\eta} \\ &= -\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|=1} f(\tilde{\zeta})\frac{\bar{z}}{1-\bar{z}\tilde{\zeta}}\,d\tilde{\xi}d\tilde{\eta} \end{aligned}$$

as is seen from the Cauchy formula.

Theorem 7 *The Neumann problem for the inhomogeneous Cauchy-Riemann equation in the unit disc*

$$w_{\bar{z}} = f \text{ in } \mathbb{D}\ ,\ \partial_\nu w = \gamma \text{ on } \partial\mathbb{D}\ ,\ w(0)=c$$

for $f\in C^\alpha(\bar{\mathbb{D}};\mathbb{C}), 0<\alpha<1, \gamma\in C(\partial\mathbb{D};\mathbb{C}), c\in\mathbb{C}$ *is solvable if and only if for* $|\,z\,|<1$

$$\begin{aligned} &\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{(1-\bar{z}\zeta)\zeta}+\frac{1}{2\pi i}\int\limits_{|\zeta|=1} f(\zeta)\frac{d\bar{\zeta}}{1-\bar{z}\zeta} \\ &\quad +\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{\bar{z}f(\zeta)}{(1-\bar{z}\zeta)^2}\,d\xi d\eta = 0\ . \end{aligned} \tag{17}$$

The unique solution then is

$$w(z) = c - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\gamma(\zeta) - \bar{\zeta} f(\zeta)) \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{|\zeta|<1} \frac{z f(\zeta)}{\zeta(\zeta - z)}\, d\xi d\eta \,. \tag{18}$$

Proof The function $\varphi = w - Tf$ satisfies

$$\varphi_{\bar{z}} = 0 \text{ in } \mathbb{D} \,,\ \partial_\nu \varphi = \gamma - z\Pi f - \bar{z} f \text{ on } \partial\mathbb{D} \,,\ \varphi(0) = c - Tf(0) \,.$$

As the properties of the Π-operator, see [10], Chapter 1, §8 and §9, guarantee $\Pi f \in C^\alpha(\bar{\mathbb{D}}; \mathbb{C})$ for $f \in C^\alpha(\bar{\mathbb{D}}; \mathbb{C})$ Theorem 4 shows

$$\varphi(z) = c - Tf(0) - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\gamma(\zeta) - \zeta \Pi f(\zeta) - \bar{\zeta} f(\zeta)) \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta}$$

if and only if

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\gamma(\zeta) - \zeta \Pi f(\zeta) - \bar{\zeta} f(\zeta)) \frac{d\zeta}{(1 - \bar{z}\zeta)\zeta} = 0 \,.$$

From

$$\begin{aligned} &\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \zeta \Pi f(\zeta) \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta} \\ &= -\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta}) \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \frac{\log(1 - z\bar{\zeta})}{(\zeta - \tilde{\zeta})^2}\, d\zeta d\tilde{\xi} d\tilde{\eta} \\ &= \frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta}) \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \frac{\log(1 - z\bar{\zeta})}{(1 - \tilde{\zeta}\bar{\zeta})^2}\, d\bar{\zeta} d\tilde{\xi} d\tilde{\eta} = 0 \,, \end{aligned}$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \Pi f(\zeta) \frac{d\zeta}{1 - \bar{z}\zeta} = -\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta}) \frac{1}{2\pi}\int\limits_{|\zeta|=1} \frac{1}{(\zeta - \tilde{\zeta})^2} \frac{d\zeta}{1 - \bar{z}\zeta}\, d\tilde{\xi} d\tilde{\eta}$$

$$= -\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta}) \partial_\zeta \frac{1}{1 - \bar{z}\zeta}\,|_{\zeta=\tilde{\zeta}}\, d\tilde{\xi} d\tilde{\eta} = -\frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta) \frac{\bar{z}}{(1 - \bar{z}\zeta)^2}\, d\xi d\eta$$

the result follows.

Theorem 8 *The problem*

$$w_{\bar{z}} = f \text{ in } \mathbb{D} \ , \ zw_z = \gamma \text{ on } \partial\mathbb{D} \ , \ w(0) = c$$

is solvable for $f \in C^\alpha(\bar{\mathbb{D}};\mathbb{C}), 0 < \alpha < 1, \gamma \in C(\partial\mathbb{D};\mathbb{C}), c \in \mathbb{C}$*, if and only if*

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{(1-\bar{z}\zeta)\zeta} + \frac{\bar{z}}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{d\xi d\eta}{(1-\bar{z}\zeta)^2} = 0 \ . \qquad (19)$$

The solution is then uniquely given as

$$w(z) = c - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \log(1-z\bar{\zeta}) \frac{d\zeta}{\zeta} - \frac{z}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{d\xi d\eta}{\zeta(\zeta - z)} \ . \quad (20)$$

Proof The function $\varphi = w - Tf$ satisfies

$$\varphi_{\bar{z}} = 0 \text{ in } \mathbb{D} \ , \ z\varphi'(z) = \gamma - z\Pi f \text{ on } \partial\mathbb{D} \ , \ \varphi(0) = c - Tf(0) \ .$$

Comparing this with the problem in the preceding proof leads to the result.

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A note on a boundary value problem for the Bitsadze equation in Wiener-type domains

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Abstract

In this article, we consider the boundary value problem

$$w_{\overline{z}\overline{z}} = F(z, w_{\overline{z}}), \quad z \in D,$$

$$\mathrm{Rew}\Big|_{\partial \mathrm{D}} = \varphi_1, \quad \mathrm{Rew}_{\overline{\mathrm{z}}}\Big|_{\partial \mathrm{D}} = \varphi_2,$$

$$\mathrm{Imw}(\mathrm{z}_0) = \mathrm{c}_1, \quad \mathrm{Imw}_{\overline{\mathrm{z}}}(\mathrm{z}_0) = \mathrm{c}_2, \quad \mathrm{z}_0 \in \overline{\mathrm{D}},$$

and discuss the conditions under which the solution exists in a Wiener-type domain $D \subset \mathbb{C}$.

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1 Introduction

The solubility of the Dirichlet problem for the Laplace equation in $D \subset \mathbb{R}^n$ with non-smooth boundary has attracted many researchers in the beginning of the twentieth century. Finally, N.Wiener has obtained the necessary and sufficient conditions (known as Wiener criterion) [11, 12] for the existence of a solution. Since then, many articles have appeared in the literature extending the Wiener test for the Dirichlet problem for nonlinear elliptic differential equations; for the relevant references, see the surveys by Keldysh [4], Landis [7], Kondrat'ev and Landis [6], Malý and Ziemer [8] and by the author [2]. Recently, the Wiener test on solubility of a boundary value problem for a first order complex differential equation with non-smooth boundary has been extended to Vekua type equations by the author and Koca [3, 5] in the class of generalized analytic functions in Vekua sense [9, 10].

Boundary value problems for higher order complex differential equations have also been treated by many authors. We will mention particularly, the paper by Akal and Begehr [1] in which the Riemann-Hilbert problem for the nonlinear second order equation

$$w_{\overline{z}\overline{z}} = H(z, w, w_z, w_{\overline{z}}, w_{z\overline{z}})$$

is discussed. This differential equation is much more general than the one we have taken. In the present article, we consider the Schwarz boundary value problem

$$w_{\overline{z}\overline{z}} = F(z, w_{\overline{z}}), \quad z \in D, \tag{1}$$

$$\mathrm{Re}w\Big|_{\partial D} = \varphi_1(z), \quad \mathrm{Re}w_{\overline{z}}\Big|_{\partial D} = \varphi_2(z), \quad z \in \partial D, \qquad \varphi_1, \varphi_2 \in C^{\alpha}(\overline{D}) \tag{2}$$

$$\mathrm{Im}w(z_0) = c_1, \quad \mathrm{Im}w_{\overline{z}}(z_0) = c_2, \quad z_0 \in \overline{D} \tag{3}$$

for the Bitsadze equation where $D \subset \mathbb{C}$ is a bounded domain with non-smooth boundary and $w : D \to \mathbb{C}$. Throughout this article we will assume that F satisfies

$$\text{(i)} \quad |F(z, s)| \le C, \tag{4}$$

$$\text{(ii)} \quad |F(z_2, s) - F(z_1, s)| \le L_1 |z_2 - z_1|^{\alpha}, \ \forall z_1, z_2 \in D, \ 0 < \alpha \le 1, \tag{5}$$

$$\text{(iii)} \quad \|F(z, s_2) - F(z, s_1)\|_{C^{1,\alpha}(D)} \le L_2 \|s_1 - s_2\|_{C^{1,\alpha}(D)}, \forall s_1, s_2 \in C^{1,\alpha}(D) \tag{6}$$

in the domain $D \times C^{1,\alpha}(\overline{D})$. Utilizing the idea given by Begehr [1] we may reduce the problem (1)-(3) into the following system of boundary value problems for the first order complex differential equations.
Problem I:

$$w_{\overline{z}} = \omega(z) \quad , \quad z \in D, \tag{7}$$

$$\mathrm{Re} w\Big|_{\partial D} = \varphi_1(z) \quad , \quad \varphi_1 \in C^{\alpha}(\partial D), \quad 0 < \alpha \leq 1, \tag{8}$$

$$\mathrm{Im} w(z_0) = c_1 \quad , \quad z_0 \in \overline{D}, \tag{9}$$

and
Problem II:

$$\omega_{\overline{z}} = F(z, \omega) \quad , \quad z \in D, \tag{10}$$

$$\mathrm{Re}\omega\Big|_{\partial D} = \varphi_2(z) \quad , \quad \varphi_2 \in C^{\alpha}(\partial D), \quad 0 < \alpha \leq 1, \tag{11}$$

$$\mathrm{Im}\omega(z_0) = c_2 \quad , \quad z_0 \in \overline{D}. \tag{12}$$

We observe that Problem I is a particular case of Problem II. The author together with K. Koca [5] has investigated Problem II for some particular choices of $F(z, \omega)$ which can also be treated for the general case of $F(z, \omega)$. For the sake of completeness we will provide some of the results of this report.

2 Existence of Solution of Problem II in Wiener - Type Domains

The complex differential equation (10) is equivalent to the real system of equations

$$\begin{aligned} u_x - v_y &= G(x, y, u, v) \\ u_y + v_x &= H(x, y, u, v). \end{aligned} \tag{13}$$

Eliminating v_x and v_y from this system, and assuming that the resulting equation does not depend on v, we get

$$\begin{aligned} &Lu = 0 \ , \ z \in D \\ &u\Big|_{\partial D} = \varphi_2 \ , \ \varphi_2 \in C^{\alpha}(\partial D) \end{aligned} \tag{14}$$

where

$$Lu := \Delta u + H_1(x, y, u, u_x, u_y).$$

We assume that there exist positive constants C_3, C_4, C_5, C_6 such that the inequalities

$$\begin{aligned}
&(a)|H_1| + |\frac{\partial H_1}{\partial x}| + |\frac{\partial H_1}{\partial u}| + (1 + \sqrt{p^2 + q^2})|\frac{\partial H_1}{\partial p}| \leq C_3(1 + \sqrt{p^2 + q^2})^2 \\
&(b)|H_1| + |\frac{\partial H_1}{\partial y}| + |\frac{\partial H_1}{\partial u}| + (1 + \sqrt{p^2 + q^2})|\frac{\partial H_1}{\partial q}| \leq C_3(1 + \sqrt{p^2 + q^2})^2 \\
&(c)|H_1(x, y, u, p, q)| \leq C_4(|p|^2 + |q|^2) + C_5 \qquad (15) \\
&(d) - C_6 \leq H_1(x, y, u, p, q) \leq 0
\end{aligned}$$

hold for all $z = x + iy \in D, \quad (u, p, q) \in \mathbb{R}^3$ where $p = u_x$ and $q = u_y$.

As we have stated previously, D is a bounded domain. So, we may choose the set $\{D_m\}_1^\infty$ of subdomains with smooth boundaries such that

$$\overline{D}_m \subset D_{m+1} \subset D, \quad m = 1, 2, \cdots, \lim_{m\to\infty} D_m = D. \qquad (16)$$

Let Φ_2 be the continuous extension of $\varphi_2 \in C^\alpha(\partial D)$ into the domain D such that $\Phi_2 \in C^\alpha(\partial D_m)$. Now we may define the classical Dirichlet problems

$$\Delta u_m + H_1(x, y, u_m, \frac{\partial u_m}{\partial x}, \frac{\partial u_m}{\partial y}) = 0, \qquad x + iy \in D_m \qquad (17)$$

$$u_m\Big|_{\partial D_m} = \Phi_2\Big|_{\partial D_m} := \Phi_{2m}, \quad m = 1, 2, \cdots$$

in domains D_m with smooth boundaries. Thus there exists a sequence $\{u_m\}_1^\infty$ of solutions of (17).

Definition 1 *(i) If*

$$\lim_{m\to\infty} u_m = u_{\varphi_2}$$

exists, then u_{φ_2} *is called the* generalized solution of (14) in Wiener sense.

(ii) Let $z_0 \in \partial D$ *be a fixed point and* u_{φ_2} *be the generalized solution of* (14) *in Wiener sense. If for each* $\varphi_2 \in C^\alpha(\partial D)$,

$$\lim_{z\to z_0} u_{\varphi_2}(z) = \varphi(z_0)$$

holds, then z_0 *is called a* regular point *in Wiener sense. Otherwise it is an* irregular point *of* ∂D.

(iii) A domain is of Wiener-type *if every point on its boundary is regular in Wiener sense.*

We need the concept of *capacity* to derive a complete necessary and sufficient characterization of regular points for the Dirichlet problem (14). Let $E, T \subset \mathbb{C}$ be Borel measurable sets, $z \in T, \xi \in E$ and $r = |z - \xi|$. Let $\mathcal{M}$ be the set of all measures defined on the σ-algebra of all subsets of E. Let us define the subset of $\mathcal{M}$ by

$$\mathcal{M}_1 := \{\mu \in \mathcal{M} : \int \int_E g(\xi, z) d\mu(\xi) \leq 1\}$$

where

$$g(\xi, z) := \left[\log \frac{\gamma}{r}\right]^s, \qquad r < \gamma.$$

We may choose s and γ such that $Lg \geq 0$.

Definition 2 *The* logarithmic (L, s)-capacity *of E with respect to T is defined by*

$$Cap_{(L,s)}E := \sup_{\mu \in \mathcal{M}_1} \mu(E).$$

To prove the existence of solutions, we will employ the arguments given in [3]. First we show that $\{u_m(z)\}_1^\infty$, is convergent:

$$\lim_{m \to \infty} u_m(z) = u_{\varphi_2}(z)$$

Next we discuss under which conditions the function $u_{\varphi_2}(z)$ coincides with $\varphi_2(z)$ on ∂D. For this part we will need

Definition 3 *Let $z_0 \in \partial D$ be a given fixed point and u be a subsolution defined in any $D' \subset D$, continuous in $\overline{D'}$ and satisfying $u(z) < 1$ for each $z \in D'$. If there exists a real valued function ψ such that*

(i) $\psi(r) > 0$ for $0 < r = |z - z_0| < r_0$ and $\lim_{r \to 0} \psi(r) = 0$

(ii) $u\Big|_{D \cap \sigma_1} \leq \psi(r)$ whenever $u\Big|_{\partial D' \cap \sigma} \leq 0$

then z_0 is called a ψ-regular point *for the boundary value problem (14), where σ and σ_1 are neighbourhoods of z_0 which may depend on s and the operator L but not on the solution and the subdomain D'.*

As in [3], it can be shown that, if z_0 is a ψ-regular point of the boundary value problem (14), then it is a regular point in Wiener sense. Thus we can prove the following theorem.

Theorem 1 *Let $Lu = 0$ be defined on a bounded domain $D \subset \mathbb{C}$ and the conditions* (15) *be satisfied. Let the solution u of $Lu = 0$ be*

continuous in $\overline{D} \setminus \{z_0\}$*, be bounded in* D *and assume the value zero on the curve* $\partial D \cap B_{R_0}(z_0)$ *where* $z_0 \in \partial D$ *is a fixed point. Let*

$$E_R := B_R(z_0) \setminus D, \qquad Cap_{(L,s)}E_R := K(R)$$

hold for $0 < R < R_0$ *and let*

$$Cap_{(L,s)}E_{4^{-m}} := K_m.$$

If the series

$$\sum_{m=m_0}^{\infty} K_m \tag{18}$$

is divergent, then $z_0 \in \partial D$ *is* ψ*-regular and there exists a constant* C_0 *such that*

$$\psi(|z - z_0|) = C_0 \exp\Bigg(- \sum_{m=m_0}^{[\log(1/|z-z_0|)]} K_m\Bigg).$$

This theorem may be proved using the techniques given for the proof of Theorem 2.3 in [3].

Corollary 1 *If the series* (18) *is divergent then the Dirichlet problem defined by* (14) *has a solution.*

To determine the imaginary part of the solution of Problem II, we use the system (13) together with the condition

$$\mathrm{Im}\omega(z_0) = v(z_0) = c_2, \qquad z_0 \in \overline{D}, \quad c_2 \in \mathbb{R}.$$

Combining u and v as

$$\omega(z) = u(x, y) + iv(x, y)$$

we find the existence of the solution of the Schwarz problem (10)-(12) in a domain of Wiener type.Thus we have obtained

Theorem 2 *The Schwarz boundary value problem defined by* (10)-(12) *in a Wiener - type domain has the unique solution*

$$\omega(z) = \mathbf{\Phi}_2(z) + T_D F(z, \omega)(z), \quad z \in D, \tag{19}$$

if $||T_D||_{C^{1,\alpha}} L_2 < 1$, where

$$T_D h(z) = -\frac{1}{\pi} \int\int_D \frac{h(\xi)}{\xi - z} d\xi d\eta, \quad \xi = \xi + i\eta,$$

and the analytic function Φ_2 satisfies the conditions

$$\mathrm{Re}\Phi_2(z) = \varphi_2(z) - \mathrm{Re}T_D F(z, \omega), \quad z \in \partial D, \tag{20}$$

$$\mathrm{Im}\Phi_2(z_0) = c_2 - \mathrm{Im}T_D F(z, \omega)(z_0), \quad , z_0 \in \overline{D}. \tag{21}$$

3 Representation of Solutions for the Schwarz Problem in Wiener-Type Domains

In this section we will show that the representation of the solution to the Schwarz problem (1)-(3) in the Hölder space $C^{1,\alpha}(\overline{D})$ using the iterated T_D operator, which was originally defined for smooth domains D, may also be extended to the Wiener-type domains. But it is trivial that the solution of Problem I is the solution of the boundary value problem for the Bitsadze equation. Problem I is uniquely solvable in Wiener - type domains. Thus we obtain

$$w(z) = \Phi_1(z) + T_D\omega(z) \tag{22}$$

where Φ_1 is an analytic function. Substituting (19) in (22), we get

$$w(z) = \Phi_1(z) + \overline{z}\Phi_2(z) + T_D^2 F(z, w_{\overline{z}}), \quad w \in C^{1,\alpha}(\overline{D}). \tag{23}$$

The analytic function Φ_1 should satisfy the conditions

$$\mathrm{Re}\Phi_1(z) = \varphi_1(z) - \mathrm{Re}\overline{z}\Phi_2(z) - \mathrm{Re}T_D^2 F(z, w_{\overline{z}})(z) \text{ on } \partial\mathrm{D}, \tag{24}$$

$$\mathrm{Im}\Phi_1(z_0) = c_1 - \mathrm{Im}\ \overline{z}_0\Phi_2(z_0) - \mathrm{Im}T_D^2 F(z, w_{\overline{z}})(z_0). \tag{25}$$

Hence we have proved

Theorem 3 *The integral representation* (23)*, for the solutions of the Schwarz problem* (1)-(3) *is also valid for the Wiener-type domains where the analytic functions* Φ_1 *and* Φ_2 *satisfy* (24)-(25) *and* (20)-(21) *respectively.*

We should note that a unique solution exists if $||T_D||_{C^{1,\alpha}(\overline{D})} L_2 < 1$.

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To elliptic boundary value problems on the upper half-plane*

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Abstract

The boundary value problems for elliptic systems of second order with leading constant coefficients are considered in a half-plane. The investigation is based on the Bitsadze formula which represents a general solution of this system through a vector-valued analytic function. The transformation of this formula is studied in Hölder spaces with weight. As a consequence explicit formulas of the solutions of the problems are received. Applications to anisotropic elasticity are also given.

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1 Elliptic systems of the second order and J- analytic functions

Let us consider the elliptic system

$$a_{11}u_{xx} + (a_{12} + a_{21})u_{xy} + a_{22}u_{yy} = 0 \tag{1}$$

with constant coefficients $a_{ij} \in \mathbb{R}^{l\times l}$ for vector-valued function $u = (u_1, \ldots, u_l)$. The ellipticity condition means that $\det a_{22} \neq 0$ and the roots of the characteristic polynomial

$$\chi = \det P, \quad P(z) = a_{11} + (a_{12} + a_{21})z + a_{22}z^2$$

are not real. Let $\nu_1, \ldots, \nu_m$ be all different roots in the half-plane $\operatorname{Im}\nu > 0$ and $l_1, \ldots, l_m$ be their multiplicities, $l_1 + \ldots + l_m = l$.
Lemma 1 *There exist matrices $b, J \in \mathbb{C}^{l\times l}$ such that*

$$\det B \neq 0, \quad B = \begin{pmatrix} b & \overline{b} \\ bJ & \overline{bJ} \end{pmatrix},$$

$$J = \operatorname{diag}(J_1, \ldots, J_m), \quad \sigma(J_i) = \{\nu_i\}, \tag{2}$$

$$a_{11}b + (a_{12} + a_{21})bJ + a_{22}bJ^2 = 0.$$

The matrix J is defined uniquely to an accuracy of similarity and can be chosen in Jordan form. The multiplicity and order of an eigenvalue ν of this matrix coincide with the multiplicity of the characteristic polynomial $\chi(x)$ and the order of the pole of the matrix-valued function $P^{-1}(z)$ at the point ν respectively.
Proof Let us consider the block matrix

$$A = \begin{pmatrix} 0 & 1 \\ a_0 & a_1 \end{pmatrix} \in \mathbb{C}^{2l\times 2l},$$

where $a_0 = -a_{22}^{-1}a_{11}$, $a_1 = -a_{22}^{-1}(a_{12} + a_{21})$ and 0 (1) means the null (unit) matrix. The identity

$$(z - A)\begin{pmatrix} 1 & 1 \\ 0 & z \end{pmatrix} = \begin{pmatrix} z & 0 \\ -a_0 & z^2 - a_1 z - a_0 \end{pmatrix}$$

shows that the characteristic polynomial of A coincides with $\chi(t)$ to the accuracy of a constant multiplier. Let us introduce an invertible

block-matrix $B = (b_{ij})_1^2 \in \mathbb{C}^{2l\times 2l}$ such that $B_{j2} = \overline{B}_{j1}$, $j = 1, 2$, and $AB = B\mathrm{diag}\,(J, \overline{J})$. From the last equality it follows that $B_{11}J = B_{12}$ and $B_{21}J = a_0B_{11} + a_1B_{21}$. Putting $B_{11} = B$, we receive (2).

Obviously we can choose the matrix J here in the Jordan form. If the pair (b_1, J_1) is the other solution of equation (2) then the matrices $\mathrm{diag}\,(J, \overline{J})$ and $\mathrm{diag}\,(J_1, \overline{J}_1)$ are similar. Since $\sigma(J) = \sigma(J_1) \subseteq \{\mathrm{Im}\, z > 0\}$, the matrixes J and J_1 have a common Jordan form and therefore are similar. The last assertion of the lemma follows from the above matrix identity. It is easily to show [16] that the condition $\det b \neq 0$ is equivalent to

$$\det\left(\int_{\mathbb{R}} P^{-1}(t)dt\right) \neq 0.$$

According to A.V.Bitsadze [3] the systems satisfying the condition $\det b \neq 0$ ($\det b = 0$) are called weakly (strongly) connected. For example the Bitsadze system

$$u_{xx} + 2eu_{xy} - u_{yy} = 0, \quad e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is strongly connected and for it

$$b = \begin{pmatrix} 1 & 0 \\ -i & 0 \end{pmatrix}, \quad J = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}.$$

The Lamé system of plane elasticity (in the isotropic case) [13]

$$\mu(u_{xx} + u_{yy}) + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\; u = 0,$$

with positive coefficients λ, μ is weakly connected and for it

$$b = \begin{pmatrix} 1 & 0 \\ i & -æ \end{pmatrix}, \quad J = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}, \quad æ = \frac{\lambda + 3\mu}{\lambda + \mu}.$$

The strongly elliptic systems by M.I.Vishik [19] are defined by the condition $P(t) > 0$, $t \in \mathbb{R}$, of positive definiteness of the matrix $P(t)$. The more restricted requirement of positive definiteness of the block matrix

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

means strong ellipticity by C. Somiliano [17]. The intermediate class corresponds to strengthened ellipticity [12]. It consists of elliptic systems (1) for which $a \geq 0$.

Let us consider the elliptic system of first order

$$\frac{\partial \phi}{\partial y} - J\frac{\partial \phi}{\partial x} = 0 \tag{3}$$

which corresponds to the Cauchy-Riemann system for $J = i$. The solution $\phi = (\phi_1, \ldots, \phi_l)$ of this system is a real analytic function and in a neighborhood of each point $z_0 = x_0 + iy_0$ it expands in a generalized Taylor series

$$\phi(z) = \sum_{k=0}^{\infty} \frac{1}{k!}[z - z_0]^k \phi^{(k)}(z_0), \quad \phi^{(k)} = \frac{\partial^k \phi}{\partial x^k},$$

where here and below we use the matrix notation $[x + iy] = x + Jy$, $x, y \in \mathbb{R}$. For Toeplitz matrix J the system (3) was firstly investigated by A. Douglis [4] in the frame of so called hypercomplex numbers. So the solutions ϕ of system (3) we call functions analytic in the Douglis sense or shortly J-analytic functions.

For these functions the generalized Cauchy type integral

$$(I\varphi)(z) = \frac{1}{\pi i}\int_{\Gamma} [t - z]^{-1}[dt]\varphi(t), \quad z \notin \Gamma,$$

plays the same role as in the usual analytic theory. In particular under the same assumptions as in the classical case $J = i$ we have the Cauchy formula $2\phi = I\phi^+$ and the Sokhotski-Plemelj formula $(I\varphi)^{\pm} = \pm\varphi + I^*\varphi$, where $I^*\varphi$ means the corresponding singular integral.

The canonical system (3) is closely connected with the second order system (1). From (2), (3) it immediately follows that the function

$$u = \operatorname{Re} b\phi \tag{4}$$

satisfies the system (1). Conversely the J-analytic function ϕ here is restored by the formula

$$\phi' = 2(b_1 u_x + c_1 u_y), \quad \begin{pmatrix} b_1 & c_1 \\ \bar{b}_1 & \bar{c}_1 \end{pmatrix} = B^{-1}, \tag{5}$$

with the block matrix B from (2).

There exists a simple connection between J-analytic and usual analytic functions. Suppose firstly $\sigma(J) = \{\nu\}$. Then a direct check shows that the transformation

$$(E\psi)(x + iy) = \sum_{j=0}^{l-1} \frac{y^j}{j!}(J - \nu)^j \psi^{(j)}(x + \nu y), \tag{6}$$

moves the analytic vector-valued functions ψ of the complex variable $x + \nu y$ into $J-$analytic functions $\phi = E\psi$. This correspondence between the functions ψ and $\phi = E\psi$ is invertible in the class C^∞. More exactly

$$\psi(\xi + i\eta) = \sum_{j=0}^{l-1} \frac{1}{j!} \left(\frac{-\eta}{\operatorname{Im} \nu} \right)^j (J - \nu)^j \frac{\partial^j \phi}{\partial x^j}(\xi + \tilde{\nu}\eta),$$

where $\tilde{\nu} = (i - \operatorname{Re} \nu)/\operatorname{Im} \nu$.

We can also consider the operation E for scalar functions ψ. In this case $E\psi$ is an $l \times l$-matrix-valued function satisfying equation (3). Thus for the scalar function $\psi(z) = z^{-1}$ we have the equality $E\psi = [z]^{-1}$.

In the general case if the vector ψ is written in block form $\psi = (\psi_1, \ldots, \psi_m)$ (analogously the block diagonal matrix $J = \operatorname{diag}(J_1, \ldots, J_m)$ in (2)), then by definition we have $(E\psi)_i = E_i\psi_i$, $i = 1, \ldots, m$, where E_i is defined by (6) with respect to $\nu = \nu_i$.

The transformation E together with (4) gives the known Bitsadze formula [3]

$$u = \operatorname{Re} bE\psi, \tag{7}$$

which describes a general solution u of (1) through an analytic function ψ.

2 Bitsadze transformation on the upper half-plane

There are two difficulties in using the transformation E. Firstly the vectors ϕ and ψ have different domains of their definition, secondly there are derivatives of the function ψ in the formula (6).

In the case of the half-plane $D = \{y > 0\}$ considered below both these obstacles are absent. Really the affine transformation $x + iy \to x + \nu y$ is invariant in D and - how will be shown below - the transformation E is invariant in the Hölder class with weight.

Let us remind the definition of these spaces. Let $C^\mu(\overline{D})$ be the usual Hölder space with the finite norm

$$|\varphi| = |\varphi|_0 + \{\varphi\}_\mu = \sup_D |\varphi(z)| + \sup_{z_1 \neq z_2} \frac{|\varphi(z_1) - \varphi(z_2)|}{|z_1 - z_2|^\mu}.$$

Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and $F \subseteq \overline{\mathbb{R}}$ be a finite set containing the point $\tau = \infty$. The family $\lambda = (\lambda_\tau, \tau \in F\}$ of real numbers is called weighted order, these orders are equipped with the natural operations. For $\lambda_\tau = \nu$, $\tau \in F$, we write $\lambda = \nu \in \mathbb{R}$. Let us put else

$$\sigma_\tau = \begin{cases} 1, & \tau \neq \infty, \\ -1, & \tau = \infty. \end{cases} \tag{8}$$

The weight function $\rho_\lambda(z)$ is a smooth function on $\mathbb{C} \setminus F$ which is everywhere different from 0 and which is identically equal to $|z - \tau|^{\lambda_\tau}$ in a neighborhood of $\tau \in F, \tau \neq \infty$, and is equal to $|z|^{\lambda_\tau}$ in a neighborhood of $\tau = \infty$. Then the space $C^\mu_\lambda(\overline{D}; F)$ is defined by the norm

$$|\varphi| = |\rho_{-\lambda}\varphi|_0 + \{\rho_{\sigma\mu-\lambda}\varphi\}_\mu.$$

The space $C^{1,\mu}_{(\lambda)}$, $0 < \lambda < 1$, of differential functions is described by the conditions $\varphi_x, \varphi_y \in C^\mu_{\lambda-\sigma}$. Below the main interest represents the space

$$\left\{\varphi \in C^{1,\mu}_{(\lambda)}(\overline{D}; F),\ \varphi(\infty) = 0\right\}, \quad 0 < \lambda < 1. \tag{9}$$

Theorem 1 *The transformation E is invertible in the class (9). Accordingly the Bitsadze formula (7) establishes an isomorphism between the solutions u of (1) and analytic vector-valued functions ψ in this class. Nevertheless $u_x(x,0) = \operatorname{Re} b\psi'(x)$, $u_y(x,0) = \operatorname{Re} bJ\psi'(x)$ and therefore $\psi'(x) = 2(b_1 u_x + c_1 u_y)(x,0)$.*

Proof For the analytic vector-valued function ψ in the class (8) we can write the Cauchy formula

$$\psi(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} (t - z)^{-1}\psi^+(t)dt$$

in the half-plane D. Since $E(t - z)^{-1} = [t - z]^{-1}$, $t \in \mathbb{R}$, we have:

$$(E\psi)(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} [t - z]^{-1}\psi^+(t)dt.$$

It may be shown analogously to [11] that the integrals (9), (10) define a boundary operators from the space $\{\psi \in C^{1,\mu}_\lambda(\mathbb{R}, F),\ \psi(\infty) = 0\}$ to the class (8). Using the Sokhotski-Plemelj formula for these integrals we have

$$2\psi^+(t_0) = \psi^+(t_0) + \frac{1}{\pi i}\int_{\mathbb{R}} (t - t_0)^{-1}\psi^+(t)dt = 2(E\psi)^+(t_0).$$

So together with (4), (5) we complete the proof.

3 The boundary value problems

Let us consider the problem

$$(p_1 u_x + p_2 u_y + p_0 u)\big|_R = g \tag{10}$$

for the equation (1) in the class (9), where the real $l \times l$-matrix-valued functions p_j are continuous on $\overline{\mathbb{R}} \setminus F$, and the right-hand function $g \in C^{\mu}_{\lambda-\sigma}(\overline{\mathbb{R}}; F)$. More exactly the functions p_j, $j = 1, 2$, permit one side limits $p_j(\tau \pm 0)$ at the points $\tau \in F$ (for $\tau = \infty$ they correspond $p_j(\mp\infty)$) and $p_j(t) - p_j(\tau \pm 0) \in C^{\mu+\varepsilon}_{\varepsilon}(\Gamma_{\tau\pm 0}, \tau)$ with some $\varepsilon > 0$ for the segment $\Gamma_{\tau\pm 0} \setminus \tau \subseteq \mathbb{R} \setminus F$ with the end point $\tau \in F$. The function p_0 then belongs to the class $C^{\mu+\varepsilon}_{\lambda-\sigma+\varepsilon}$, $\varepsilon > 0$.

Because of Theorem 1 this problem reduces equivalently to the problem

$$\operatorname{Re}(G\psi' + G_0\psi)\big|_R = g \tag{11}$$

with the coefficients $G = p_1 b + p_2 bJ$, $G_0 = p_0 b$ for the analytic vector-valued function $\psi(z)$. For the latter one can use the result of the classical theory [9, 5, 18]. In particular with the help of the canonical function $X(z)$ its solution can represented by an explicit formula.

Using the Bitsadze transformation we can reformulate these results to the original problem (10). By definition this problem is of normal type if the matrix $G = p_1 b + p_2 bJ$ satisfies the conditions

$$\det G(t) \neq 0, \quad t \in \mathbb{R} \setminus F; \quad \det G(\tau \pm 0) \neq 0, \quad \tau \in F.$$

In this case one can introduce the increment

$$\arg(\det G)\big|_{\mathbb{R}} = \sum_{\tau \in F} \mp \arg \det G(\tau \pm 0), \tag{12}$$

which does not depend on the choice of arg and the family of invertible matrixes

$$Q_\tau = (G\overline{G}^{-1})(\tau + 0)(\overline{G}G^{-1})(\tau - 0), \quad \tau \in F,$$

the eigenvalues $w \in \sigma(Q_\tau)$ of which lie on the unit circle $|w| = 1$.

With the matrix Q of this type we can connect the piecewise constant function

$$\chi(\cdot, Q) = \sum_{w \in \sigma(Q)} l(w) \left(\left[\cdot - \frac{\arg w}{2\pi} \right] + \frac{\arg w}{2\pi} \right), \tag{13}$$

which does not depend on the choice of arg . Here the notation [] means the integer part of a number and $l(w)$ is the multiplicity of the eigenvalue w.

Theorem 2 *The problem* (1), (10) *in the class* (9) *is Fredholm if and only if it is of normal type and* $e^{2\pi i\lambda_\tau} \notin \sigma(Q_\tau)$, $\tau \in F$. *Its index is given by the formula*

$$æ = -\frac{1}{\pi} \arg\det G\big|_{\mathbb{R}} - \sum_{\tau\in F} \chi(\lambda_\tau, Q_\tau) - l. \tag{14}$$

From the definition (13) it follows that $\chi(\delta, Q) = (2\pi)^{-1} \arg\det Q+$ integer. So the right-hand part of (14) is an integer.

Proof The problem (1), (10) is equivalent to (11) in the class (9). So it is sufficient to prove the theorem for the problem (11).

The operator of differentiation $\psi \to \psi'$ is invertible from the class (9) to $C^\mu_{\lambda-\sigma}$. For $\psi_1 = \psi' \in C^\mu_{\lambda-\sigma}$ let us put $T\psi_1 = \operatorname{Re} G_0\psi^+$. Under accepted assumptions with respect to p_j this operator is bounded $C^\mu_{\lambda-\sigma}(\overline{D}; \overline{F}) \to C^{\mu+\varepsilon}_{\lambda-\sigma+\varepsilon}(\mathbb{R}; F)$ with some $\varepsilon > 0$. Since the embedding $C^{\mu+\varepsilon}_{\nu+\varepsilon} \subseteq C^\mu_\nu$ is compact the operator T is also compact $C^\mu_{\lambda-\sigma}(\overline{D}; F) \to C^\mu_{\lambda-\sigma}(\mathbb{R}; F)$. So the problem (11) in the class (9) is Fredholm equivalent to the problem

$$\operatorname{Re} G\psi_1 = g_1 \tag{15}$$

in the class $C^\mu_{\lambda-\sigma}$ and their indices coincide.

With the help of the substitution $\tilde\psi_1(\tilde z) = \psi_1(z)$, $\tilde z = (z-i)/(z+i)$, problem (15) reduces to the analogue one

$$\operatorname{Re} \tilde G\tilde\psi_1 = \tilde g_1 \tag{15$^\sim$}$$

in the unit circle $\tilde D = \{|\tilde z| < 1\}$. By this substitution the space $C^\mu_{\lambda-\sigma}(D; F)$ transforms into $C^\mu_{\lambda-\sigma}(\tilde D; \tilde F)$ and $\tau = \infty$ corresponds to $\tilde\tau = 1$.

The arguments from the proof of theorem 3 in [14] for the scalar case are also applicable to the problem (15$^\sim$) with the matrix coefficient $\tilde G$. So the Fredholm condition of this problem is given by (12) and its index formula in the class $C^\mu_{\lambda-\sigma}$ is

$$\tilde æ = l - \frac{1}{\pi}\arg\det \tilde G\big|_{\partial\tilde D} - \sum_{\tau\in F} \chi(\lambda_\tau - \sigma_\tau, Q_\tau) - ml,$$

where m is the number of elements of F. Taking (8) and the relation $\chi(\delta + 1) = \chi(\delta) + l$ into account we can rewrite this equality in the form (14). That completes the proof.

4 The problems with constant coefficients

Let us consider the problem

$$(p_1 u_x + p_2 u_y)\big|_{\partial D} = g, \quad p_j \in \mathbb{R}^{l\times l}, \tag{16}$$

for which the matrix $G = p_1 b + p_2 bJ \in \mathbb{C}^{l\times l}$. In this case we can essentially complete Theorem 2.

Theorem 3 *If* $\det G = 0$, *then the homogeneous problem* (1), (16) *in the class* (9) *has an infinite number of linear independent solutions. For example if* $G\eta = 0$ *for some* $\eta \in \mathbb{C}^l$, $\eta \neq 0$, *then the function* $u = \operatorname{Re} b(E\psi_0)\eta$ *is a solution of the homogeneous problem (1), (15) for each scalar analytic function* ψ_0 *belonging to the class (9).*

If $\det G \neq 0$, *then the condition*

$$\int_{\mathbb{R}} g(t)dt = 0 \tag{17}$$

is necessary and sufficient for the one-to-one solvability of the problem in the considered class and its solution is given by the formula

$$u(x,y) = \operatorname{Re}\frac{1}{\pi}\int_{\mathbb{R}} b(t-x-Jy)^{-1}G^{-1}f(t)dt, \quad f(t) = \int_{-\infty}^{t} g(s)ds. \tag{18}$$

Note that by virtue of (17) the function $f \in C^{1,\mu}_{(\lambda)}$ has values $f(\pm\infty) = 0$ and therefore in (18) the integral exists.

Proof In the considered case the problem (11) transforms to

$$\operatorname{Re} G\psi' = g. \tag{19}$$

If $\det G = 0$, $G\eta = 0$ then the function $\psi(z) = \psi_0(z)\eta$ with scalar and analytic ψ_0 satisfies the homogeneous problem (19).

Let $\det G \neq 0$. Since the analytic function $G\psi'$ belongs to $C^{\mu}_{\lambda-\sigma}(D;F)$, $0 < \lambda < 1$, we can write the Cauchy formula for this problem in the half-plane D. In particular the condition (17) is necessary for the solvability of the problem (19) and so for the solvability of (16). From this condition it follows that $f(+\infty) = f(-\infty) = 0$ and we can assume $\psi(\infty) = f(\infty) = 0$. Using the Schwarz formula [9] for the function $G\psi$, we get the representation

$$\psi(z) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{G^{-1}f(t)dt}{t-z}.$$

By virtue of (7) and the equality $E(t-z)^{-1} = [t-z]^{-1}$, $t \in \mathbb{R}$, we come to the formula (18).

For the given system (1) we can always find matrices $p_j \in \mathbb{R}^{l\times l}$ such that $\det(p_1 b_1 + p_2 bJ) \neq 0$ i.e. matrices for which problem (16) is correct. For example in the case of the Bitsadze system it is sufficient to put $p_1 = p$, $p_2 = 1$, where $p \neq s + te$ for each $s, t \in \mathbb{R}$. On the other hand for each matrices $p_1, p_2 \in \mathbb{R}^{l\times l}$ there always exists the elliptic system (1) for which $\det(p_1 b + p_2 bJ) = 0$. So an "universal" problem of type (16) which is correct for each elliptic system (1) does not exist. But for nonlocal problems the situation is quite different.

Theorem 4 *The problem of Carleman type*

$$u(x,0) + u(-x,0) = f(x), \quad u_y(x,0) - u_y(-x,0) = g'(x) \tag{20}$$

is uniquely solvable for each elliptic system (1) *in the class* (9) *and its solution is given by the formula*

$$u(x,y) = \operatorname{Re} \frac{2}{\pi} \int_{\mathbb{R}} b(t - x - Jy)^{-1}(b_1 f + c_1 g)(t)dt, \tag{21}$$

where b_1 *and* c_1 *are defined in* (5).

Proof According to Theorem 1 we can represent the equalities (20) in the form

$$\operatorname{Re} b\{\psi(x) + \psi(-x)\} = f, \quad \operatorname{Re} bJ\{\psi'(x) - \psi'(-x)\} = g'(x).$$

Integrating the last relation and using notation (5) we have

$$\psi(x) + \psi(-x) = \tilde{f}(x), \quad \tilde{f} = 2(b_1 f + c_1 g).$$

Putting

$$\tilde{\psi}(z) = \begin{cases} \psi(z), & \operatorname{Im} z > 0, \\ -\psi(-z), & \operatorname{Im} z, > 0, \end{cases}$$

we can rewrite the above boundary condition in the linear conjugation form $\tilde{\psi}^+ - \tilde{\psi}^- = f$. So based on the Sokhotski-Plemelj formula we can give the solution of this problem in the explicit form

$$\psi(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(t)dt}{t - z}, \quad z \in D.$$

Substituting it into (7) we get (21).

The boundary condition (16) we can integrate and write in a simpler form. Let us introduce the conjugate function v to the solution u of the equation (1) by the rule

$$v_x = a_{21}u_x + a_{22}u_y, \quad v_y = -(a_{11}u_x + a_{12}u_y).$$

Substituting the representation (4) we get

$$v = \operatorname{Re} cE\psi, \quad c = a_{21}b + a_{22}bJ. \tag{22}$$

Since $u_y = a_{22}^{-1}(v_x - a_{21}u_x)$, from (15) we can move to the boundary condition

$$(pu + bv)\big|_R = f, \quad f(t) = \int_{-\infty}^{t} g(s)ds,$$

with the coefficients $p = p_1 - p_2a_{22}^{-1}a_{21}$, $q = p_2a_{22}^{-1}$. In this case $G = pb + qc$. For $G = b$ ($G = c$) we have the first (second) boundary value problem i.e. the Dirichlet (Neumann) problem.

It is convenient to reformulate Theorem 4 for these problems.

Theorem 5 (a) *Let the system* (1) *be weakly connected, i.e.* $\det b \neq 0$. *Then the Dirichlet problem is uniquely solvable in the class* (9) *and its solution* u *and the conjugate function* v *are defined by the formulas*

$$u(x,y) = \operatorname{Re}\frac{1}{\pi i}\int_{\mathbb{R}} b(t - x - Jy)^{-1}b^{-1}f(t)dt,$$

$$v(x,y) = \operatorname{Re}\frac{1}{\pi i}\int_{\mathbb{R}} c(t - x - Jy)^{-1}b^{-1}f(t)dt.$$

If this system is strongly connected then the homogeneous problem has an infinite number of linear independent solutions.

(b) *In notations (20) let* $\det c \neq 0$. *Then the Neumann problem is uniquely solvable in the class (9) and its solution* u *and the conjugate function* v *are defined by the formulas*

$$u(x,y) = \operatorname{Re}\frac{1}{\pi i}\int_{\mathbb{R}} b(t - x - Jy)^{-1}c^{-1}f(t)dt,$$

$$v(x,y) = \operatorname{Re}\frac{1}{\pi i}\int_{\mathbb{R}} c(t - x - Jy)^{-1}c^{-1}f(t)dt.$$

If $\det c = 0$, *then the homogeneous problem has an infinite number of linearly independent solutions.*

This effect connected with the Dirichlet problem was firstly opened by A.V. Bitsadze [2] in 1948 (for his system in the unit circle). In particular for strongly elliptic systems the Dirichlet problem is always uniquely solvable. But for the Neumann problem this can not be asserted even for strengthened elliptic systems. For example the system

$$u_{xx} - (q + q^{\mathrm{T}})u_{xy} + u_{yy} = 0,$$

where the matrix $q \in \mathbb{R}^{l\times l}$ is orthogonal and $\det(1\pm q) \neq 0$, is strengthened elliptic but the correspondence matrix $c = qb - bJ$ is not invertible.

It may be shown [16] that the condition $\det c \neq 0$ is fulfilled for the strengthened system (1) with the additional requirement $\operatorname{rang} a \geq 2l - 1$, $a = \{a_{ij}\}_1^2$.

5 Application to anisotropic plane elasticity

A plane elastic medium is characterized by the displacement vector $u = (u_1, u_2)$ and by the stress and deformation tensors

$$\sigma = \begin{pmatrix} \sigma_1 & \sigma_3 \\ \sigma_3 & \sigma_2 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 & \varepsilon_3 \\ \varepsilon_3 & \varepsilon_2 \end{pmatrix},$$

where $\varepsilon_1 = u_{1x}$, $\varepsilon_2 = u_{2y}$, $2\varepsilon_3 = u_{1y} + u_{2x}$. The vectors $\tilde{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $\tilde{\varepsilon} = (\varepsilon_1, \varepsilon_2, 2\varepsilon_3)$ are connected by the Hooke law i.e. by a linear relation [10, 8]

$$\tilde{\sigma} = \alpha\tilde{\varepsilon}, \quad \alpha = \begin{pmatrix} \alpha_1 & \alpha_4 & \alpha_5 \\ \alpha_4 & \alpha_2 & \alpha_6 \\ \alpha_5 & \alpha_6 & \alpha_3 \end{pmatrix} > 0.$$

If external forces are absent then the equilibrium equations have the form $\sigma_{(1)x} + \sigma_{(2)y} = 0$, where $\sigma_{(j)}$ means $j-$ th column of the matrix σ. Using the Hooke law we receive the Lamé system (1) for the replacement vector u with the coefficients defined by the matrix

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_6 & \alpha_6 & \alpha_4 \\ \alpha_6 & \alpha_3 & \alpha_3 & \alpha_5 \\ \alpha_6 & \alpha_3 & \alpha_3 & \alpha_5 \\ \alpha_4 & \alpha_5 & \alpha_5 & \alpha_2 \end{pmatrix}.$$

This system is strengthened elliptic and $\operatorname{rang} a = 3$. In terms of the conjugate function v we can write the expression of the stress tensor σ in the form $\sigma_{(1)} = -v_y$, $\sigma_{(2)} = v_x$.

So the Neumann problem on the upper half-plane corresponds to setting up the normal component $\sigma_{(2)}$ of the tensor σ. The solutions of the Dirichlet and Neumann problems are given by the above formulae (D) and (N) respectively.

For the matrix J taken in Jordan form only the two cases

$$\text{(i) } J = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}, \quad \nu_1 \neq \nu_2; \quad \text{(ii) } J = \begin{pmatrix} \nu & 1 \\ 0 & \nu \end{pmatrix}$$

are possible. According to these cases the matrices b and c can be described explicitly [15].

Let scalar polynomials p_i, q_i, $i = 1, 2, 3$, be defined by the equalities

$$P = \begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix}, \quad Q = \begin{pmatrix} p_2 & -p_3 \\ -p_3 & p_1 \end{pmatrix},$$

$$R(z) = (a_{21} + a_{22}z)Q(z) = \begin{pmatrix} zq_2 & q_1 \\ -q_2 & q_3 \end{pmatrix}.$$

(i) *The columns of the matrices b and c are determined by the equalities*

$$b_{(i)} = k_i \begin{cases} Q_{(1)}(\nu_i) \\ Q_{(2)}(\nu_i) \end{cases}, c_{(i)} = k_i \begin{cases} R_{(1)}(\nu_i) \\ R_{(2)}(\nu_i) \end{cases}, \quad i = 1, 2,$$

where $k_i \neq 0$ and the upper (lower) equality is chosen under the condition $q_2(\nu_i) \neq 0$ ($q_1(\nu_i) \neq 0$).

(ii) *The columns of the matrix b are given by the equalities*

$$b_{(1)} = k_1 Q_{(1)}(\nu), \quad b_{(2)} = k_1 Q'_{(1)}(\nu) + k_2 Q_{(1)}(\nu)$$

with arbitrary $k_j \in \mathbb{C}$, $k_1 \neq 0$. The matrix c is defined analogously with respect to $R(z)$.

In the orthotropic medium when $\alpha_5 = \alpha_6 = 0$, the characteristic equation $\det P = 0$ is biquadratic and its roots are described explicitly. In particular for an isotropic medium we have the case (ii) with $\nu = i$ and expressions

$$b = \begin{pmatrix} 1 & 0 \\ i & -æ \end{pmatrix}, \quad c = \mu \begin{pmatrix} 2i & æ - 1 \\ 2 & i(æ + 2) \end{pmatrix}, \quad æ = \frac{\lambda + 3\mu}{\lambda + \mu}.$$

With respect to another function theoretical approaches to orthotropic elasticity see [1, 6, 7].

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Snapshots in Applied Complex Analysis

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Middle East Technical University
Ankara, Turkey

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Preface

The intension of this workshop was as well to instruct newcomers in the field of applied complex analysis about recent trends as to give them the opportunity to meet some of the experts. Among the younger participants were Algerians, Germans, Chinese, Turkish, Tajik, and Vietnamese. The experts came from Armenia, Belarus, China, Germany, France/Poland, Georgian, India, Italy, Russia, Tanzania, Turkey, USA. The workshop was financially supported by the Middle East Technical University in Ankara, the Scientific and Technical Research Council of Turkey and the Ankara Branch of the Turkish Mathematical Society. Their generous support is gratefully acknowledged. Without it the character of the workshop would have been completely different as most of the young generation and many experts in particular from the former SU and countries with weak economy would not have participated.

At the workshop some series of lectures and about 25 talks were given and the selection of manuscript in this volume reflects the range of topics but not the familiar and stimulating atmosphere at the meeting. The comfortable hosting at the guest house and dormitories of the METU, the kind care taking of the local organizers and their helpers and the homogeneous composition of the 30 participants had created a friendly background for an exciting workshop.

The topics vary from complex ordinary differential equations and regular holomorphic systems to complex partial differential equations, from boundary value problems of Riemann, Haseman and Riemann-Hilbert type to Hele Shaw flows and porosity of cancellous bones, from domain perturbation problems to optimization of fixed point methods, from quaternionic to Clifford analysis.

Growth estimations in a certain angular domain for entire finite order solutions to inhomogeneous linear complex ordinary differential equations with entire coefficients are provided under wider conditions as in the past. Motivated by investigations of the geometry of a-points of a meromorphic function the closedness of a- and b-points of an arbitrary polynomial for different a and b are measured and the sum of these measures for several pairs of different points estimated from above where the upper bound is independent of the number of pairs. Combining methods from singularity theory, complex analysis, and ordinary differential equations holonomic Picard-Fuchs systems of differential equations regular singular along hypersurfaces are studied. As an introduction to the theory of boundary value problems for complex partial differential equations of arbitrary order the basic problems of Schwarz, Dirichlet, and Neumann are solved for the inhomogeneous Cauchy-Riemann equation. The Schwarz problem is considered in Wiener domains for some particular quasilinear Bitsadze equation.

Eliptic systems of 2n second order equations with constant leading coefficients are treated in the upper half plane on the basis of the Bitsadze representation and the results applied to anisotropic plane elasticity. Three versions of the modified doubly-periodic second fundamental complete plane strain problem with relative displacements are formulated and solved via the Sherman transform the kernel function of which is given by the Weierstraß ζ-function.

More advanced boundary value problems in complex analysis are the Riemann-Hilbert and the Riemann problem, the first being a generalization of the Schwarz problem and in particular situations related to the latter. The Riemann problem is a special form as well of the general $\mathbb{R}$-linear conjugation problem as of the Haseman problem where also boundary shifts are involved. An $\mathbb{R}$-linear conjugation problem for elliptic functions in a disc is solved in form of Eisenstein series via a transformation to a functional equation. The result is applied to calculate the effective conductivity of a composite material with circular inclusions. A general formulation of the Riemann-Hilbert problem on almost complex manifolds is given within the theory of geometric and topological aspects of holomorphic curves in a loop spaces. Reduction to related problems for analytic functions is used to treat the Haseman boundary value problem for bianalytic functions.

The celebrated Hele-Shaw moving boundary value problem is considered in a doubly connected domain in decomposing it in a Schwarz boundary value problem and an abstract Cauchy-Kowalewski problem. Local existence and uniqueness of the classical solution is proved. The dependence of the porosity of cancellous bones on the measured ultrasonic wave is numerically computed. Here the Biot system of compressional wave equations for a dispersive dissipative fluid-saturated porous medium in the time domain is applied.

With functional analytic and potential theoretic methods domain perturbation problems for boundary value problems for homogeneous elliptic equations are investigated. In particular the Dirichlet and the Neumann problems for the Laplace equation are considered in the n-dimensional Euclidean space. As is well-known the solution of the Picard-Lindelöf integral equation is locally uniquely solvable. This situation often occurs when initial or boundary value problems are solved iteratively. Here the problem is studied how this "local" can be optimized.

Complex analytic methods are not restricted to plane problems. Quaternionic, octonionic, and Clifford analysis are theories efficiently used in mathematical physics where complex analytic methods are applied. Recently higher order Helmholtz operators and their factors were considered in quaternionic analysis as were higher order Laplace operators and their factors in Clifford analysis. On the basis of the quaternionic Stokes formula orthogonal decompositions of the Lebesgue space of square integrable quaternionic-valued functions are given with respect to the poly-α-hyperholomorphic and the polymetaharmonic functions. Higher order Cauchy-Pompeiu representations in Clifford analysis are known for powers of the Dirac operator D and the Laplace operator $\Delta = D\bar{D}$. Here such integral representations are developed for arbitrary operators $D^m\bar{D}^n$ for $m, n \in \mathbb{N}$. Also an addendum to those representation formulas in a universal Clifford algebra in case of an even-dimensional Eulidean space of dimension n is given with respect to D^k for $n \leq k$. Higher order Cauchy-Pompeiu representations for functions of $n, 1 \leq n$, complex variables are available in particular for polydomains. A general formula is given in connection with the operators $\partial_{z_\kappa}^{k_\kappa}\partial_{\bar{z}_\lambda}^{l_\lambda}, k_\kappa, l_\lambda \in \mathbb{N}, 1 \leq \kappa, \lambda \leq n$. Properties of the polyharmonic Green function for the upper half of the complex plane are studied.

The manuscripts were composed by Mr Kerem Kaskalouglu, from the Department of Mathematics of the METU and by Mrs. Barbara G. Wengel at the I. Mathematical Institute of FU Berlin. Thanks are due to both of them. The editors are also grateful to Professor A. Anastassiou for having offered to publish this volume in his new journal JAFA as first issues. They wish JAFA a properous future.

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Modified doubly-periodic second fundamental complete plane strain problem with relative displacements

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Abstract

Three new formulations and solutions of the modified doubly-periodic second fundamental complete plane strain problem with relative displacements are posed and given by complex variable method.

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Keywords: plane strain problem, doubly periodic plane strain, elastic body with cylindric holes, Sherman transform, Kolosov function, elliptic function, Plemelj formulae

1 Formulations

A new formulation of the second fundamental problem with relative displacements has been posed for the plane (two-dimensional) elasticity problem in the non-periodic case by [8]. In this paper we attempt to pose three new formulations of the modified doubly-periodic second fundamental complete plane strain (CPS) (a particular three-dimensional) problem with relative displacements based on the author's former studies [4]-[7]. In this case, a three-dimensional elastic body with cylindrical holes is considered. It will be assumed that there exists a system of m holes with smooth and non-intersecting boundaries $\mathcal{L}_j \equiv L_j \ (mod\ 2\omega_1, 2\omega_2)$ $(j = 0, 1, \cdots, m-1)$ inside of each periodic parallelogram on the $x_1,\ x_2$ plane, and L_j $(j = 0, 1, \cdots, m-1)$, the boundaries of the holes in the doubly periodic fundamental parallelogram P_{00}, are denoted by $L = \bigcup_{j=0}^{m-1} L_j$. The elastic regions located on the left-hand and right-hand sides of L in P_{00} on the $x_1,\ x_2$ plane are denoted by S_0 and S_j^-, respectively (see Figure). The origin will be chosen inside S_0. The so-called second fundamental CPS problem with relative displacement is to determine the elastic equilibrium, when the displacements $g_j(t) = u_j(t) + iv_j(t)$ $(j = 0, 1, \cdots, m-1)$ given on L_j are relative to certain rigid motions which are different to each other for different contours. It means the undetermined real displacements should be $g_j(t) + i\alpha_j t + c_j'$ $(j = 0, 1, \cdots, m-1)$, here α_j $(j = 0, 1, \cdots, m-1)$ will be undetermined real constants, while the c_j' are complex. And the displacement $w(t)$ on L_j in x_3-direction with cyclic increments w_k $(k = 1, 2)$ is also given. The strain e_3 is *constant*.

In addition, for the unique existence of the solution in the present case, the external resultant principal vector $X_{1j} + iX_{2j}$ and the moment M_j of the traction exerted on L_j must be given. To be specific, we assume $\alpha_0 = 0$.

For the complete plane strain problem, the stress and displacement components will be expressed by the complex stress functions $\phi(z)$, $\psi(z)$, $\Phi(z) = \phi'(z)$ and $\Psi(z) = \psi'(z)$ as, see [9],[3],

$$\sigma_1 + \sigma_2 = 4 \text{ Re } \{\Phi(z)\}, \tag{1}$$

$$\sigma_2 - \sigma_1 + 2i\tau_{12} = 2[\overline{z}\Phi'(z) + \Psi(z)], \tag{2}$$

$$2\mu_z(u + iv) = \kappa_z\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} - 2\mu_z e_3 \nu_z z, \tag{3}$$

$$\sigma_3 = 2\mu_z(1+\nu_z)e_3 + \nu_z(\sigma_1+\sigma_2), \tag{4}$$

$$w = e_3 x_3, \tag{5}$$

where

$$\mu_z = \begin{cases} \mu^+, z \in S^+, \\ \mu^-, z \in S^-, \end{cases}, \quad \nu_z = \begin{cases} \nu^+, z \in S^+, \\ \nu^-, z \in S^-, \end{cases}$$

$$\mu^\pm = \frac{E^\pm}{2(1+\nu^\pm)},$$

u, v, and w are the displacement components in the x_1, x_2, x_3 directions. $\kappa_z = 3 - 4\nu_z$. $E^\pm$ (> 0), $\nu^\pm$ $(0 < \nu^\pm < \frac{1}{2})$ are the elastic Young's modulus and the Poisson ratio of the materials in $S^\pm$, respectively.

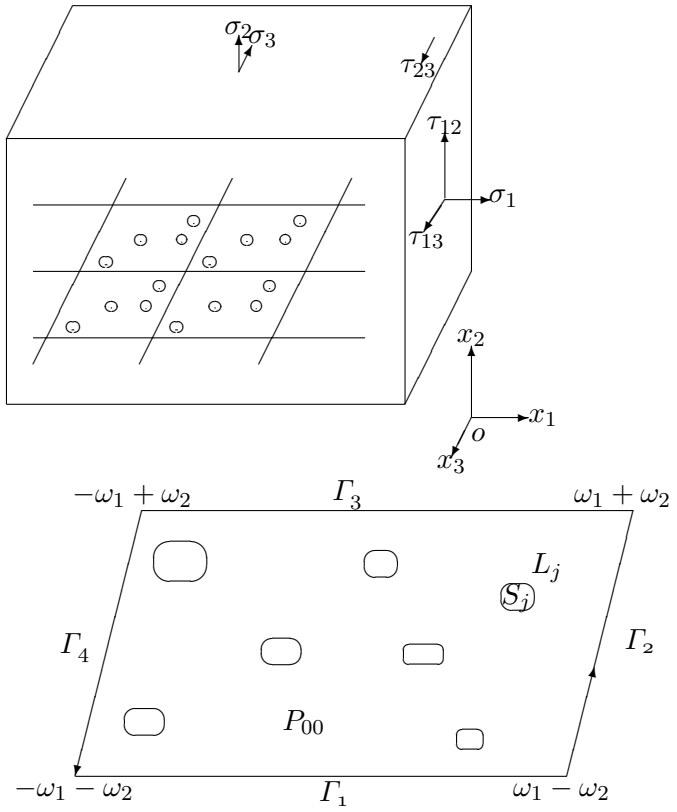


Figure: Model of an elastic body with a doubly-periodic set of holes

The longitudinal stress and displacement components will be expressed by the third complex stress function $F(z)$ which is called the

454,425

complex *torsion* function, see [9], as, see [3],

$$\tau_{13} - i\tau_{23} = 2\mu_z F'(z), \tag{6}$$

$$w = F(z) + \overline{F(z)}. \tag{7}$$

The complex stress functions $\phi(z)$ and $\psi(z)$ need not always be single-valued in a multiply connected region, although of course the components of the stress and displacement are single-valued. In order to separate the multi-valued parts of $\phi(z)$ and $\psi(z)$, we need to construct the Kolosov functions in this case,

$$\phi(z) = \frac{-1}{2\pi(\kappa+1)} \sum_{j=0}^{m-1} (X_{1j} + iX_{2j}) \log \sigma(z - z_j) + \phi_0(z), \tag{8}$$

$$\psi(z) = \frac{\kappa}{2\pi(\kappa+1)} \sum_{j=0}^{m-1} (X_{1j} - iX_{2j}) \log \sigma(z - z_j) + \psi_0(z), \tag{9}$$

where z_j are points arbitrarily situated inside S_j, $\phi_0(z)$ and $\psi_0(z)$ are holomorphic functions and hence single-valued in S_0. $\sigma(z)$ is the Weierstrass *sigma* function, see [1],

$$\sigma(z) = z \prod_{m,n}{}' \left(1 - \frac{z}{\Omega_{mn}}\right) exp(\frac{z}{\Omega_{mn}} + \frac{z^2}{2\Omega_{mn}}).$$

We have the derivatives of $\phi(z)$ and $\psi(z)$ as follows,

$$\Phi(z) = \frac{-1}{2\pi(\kappa+1)} \sum_{j=0}^{m-1} (X_{1j} + iX_{2j}) \zeta(z - z_j) + \Phi_0(z), \tag{10}$$

$$\Psi(z) = \frac{\kappa}{2\pi(\kappa+1)} \sum_{j=0}^{m-1} (X_{1j} - iX_{2j}) \zeta(z - z_j) + \Psi_0(z). \tag{11}$$

Here $\zeta(z)$ is the Weierstrass *zeta* function, see [1], namely

$$\zeta(z) = \frac{1}{z} + \sum_{m,n}{}' \left\{ \frac{1}{z - \Omega_{mn}} + \frac{1}{\Omega_{mn}} + \frac{z}{\Omega_{mn}^2} \right\},$$

where $\Omega_{mn} = 2m\omega_1 + 2n\omega_2$, the prime of $\sum$ above indicates that m and n will not be zero simultaneously. In fact, $\zeta(z)$ is a doubly quasi-periodic function with cyclic increments $2\eta_k$, $k = 1, 2$, such that

$$\zeta(z + 2\omega_k) = \zeta(z) + 2\eta_k, \tag{12}$$

$$\eta_k = \zeta(\omega_k), k = 1, 2,$$
$$\omega_2\eta_1 - \omega_1\eta_2 = \frac{\pi}{2}i. \tag{13}$$

From (3), (8) and (9) and the known relationship between displacements and complex stress functions one has immediately

$$\begin{aligned} 2\mu(u+iv) \;=\; & -\frac{\kappa}{\pi(1+\kappa)}\sum_{j=1}^{m}(X_{1j}+iX_{2j})log|\sigma(z-z_j)| \\ & +\; \frac{z}{2\pi(1+\kappa)}\sum_{j=1}^{m}(X_{1j}+iX_{2j})\overline{log\zeta(z-z_j)} \\ & +\; \kappa\phi_0(z) - z\overline{\phi_0'(z)} - \overline{\psi_0(z)} - 2\mu e_3\nu z. \end{aligned} \tag{14}$$

Then we have the displacement conditions on the boundaries of the holes

$$\kappa\phi_0(t) - t\overline{\phi_0'(t)} - \overline{\psi_0(t)}] = 2\mu g^*(t) + i\beta_j t + c_j, t \in L_j, \tag{15}$$

where

$$\begin{aligned} 2\mu g^*(t) \;=\; & 2\mu\left[g(t) + e_3\nu t\right] + \frac{\kappa}{\pi(1+\kappa)}\sum_{j=1}^{m}(X_{1j}+iX_{2j})log|\sigma(z-z_j)| \\ & -\frac{t}{2\pi(1+\kappa)}\sum_{j=1}^{m}(X_{1j}+iX_{2j})\overline{log\zeta(z-z_j)}, \end{aligned} \tag{16}$$

$$g(t) = g_j(t) + i\alpha_j t + c_j', t \in L_j,$$

$\beta_j = 2\mu\alpha_j,\ c_j = 2\mu c_j',\ j = 0, 1, \cdots, m-1.$

We can simplify the formula of the moment conditions. In fact, the moment of the traction exerted on an arc $\widehat{AB}$ (from the positive side) can be expressed as [9]

$$M_{\widehat{AB}} = \ \mathrm{Re}\ \left[\chi(z) - z\psi(z) + |z|^2\phi'(z)\right]_A^B, \tag{17}$$

when the arc $\widehat{AB}$ is chosen as a closed contour C in S_0. Because $\phi'(z)$ is single-valued in S_0, we have immediately,

$$\begin{aligned} M_C \;&=\; \mathrm{Re}\ \left[\chi(z) - z\psi(z)\right]_C \\ &=\; \mathrm{Re}\ \int_C \frac{d}{dz}\left[\chi(z) - z\psi(z)\right]dz \\ &=\; -\ \mathrm{Re}\ \int_C z\psi'(z)dz. \end{aligned} \tag{18}$$

We are interested in the moment of the traction exerted on L_j. Let $C = L_j$ $(j = 0, 1, \cdots, m-1)$, in equation (18). Substituting (11) into equation (18), we get

$$\mathrm{Re}\left\{\int_{L_j} t\left[\frac{\kappa}{2\pi(1+\kappa)}\sum_{j=0}^{m-1}(X_{1j}-iX_{2j})\zeta(t-z_j)+\psi_0'(t)\right]dt\right\} = -M_j. \tag{19}$$

Because $\psi_0(z)$ is single-valued in the region S_0, by integration by parts, hence,

$$\int_{L_j} t\psi_0'(t)dt = -\int_{L_j}\psi_0(t)dt.$$

Furthermore, from (19),

$$\mathrm{Re}\int_{L_j}\psi_0(t)dt = M_j + \mathrm{Re}\left[\frac{\kappa}{2\pi(1+\kappa)}\sum_{j=0}^{m-1}\int_{L_j}(X_{1j}-iX_{2j})t\zeta(t-z_j)dt\right]. \tag{20}$$

Because as known

$$\zeta(z-z_j) = \frac{1}{z-z_j} + \zeta_0(z, z_j),$$

where $\zeta_0(z, z_j)$ is a holomorphic function, one gets

$$\mathrm{Re}\int_{L_j}\psi_0(t)dt = M_j + \frac{\kappa}{\kappa+1}Im[z_j(X_{1j}-iX_{2j})]. \tag{21}$$

Thus, we may give three formulations of the modified doubly-periodic second fundamental CPS problem with relative displacements, say, (**MDPP1**), (**MDPP2**) and (**MDPP3**), respectively.

The first formulation is, except the above formulated conditions, given the external resultant principal vectors F_k on Γ_k $(k = 1, 2)$, to find the state of elastic equilibrium. In this case, proceeding in an analogous manner, one finds, in the former notation, the boundary conditions

$$(\mathbf{MDPP1})\quad \begin{cases} \kappa\phi_0(t) - t\overline{\phi_0'(t)} - \overline{\psi_0(t)} = 2\mu g^*(t) + i\beta_j t + c_j, t \in L_j, \\ \left[\phi_0(z) + z\overline{\phi_0'(z)} + \overline{\psi_0(z)}\right]_{\Gamma_k} = iF_k, k = 1, 2, \\ \mathrm{Re}\left[\int_{L_j}\psi_0(t)dt\right] = M_j, j = 0, 1, \cdots, m-1, \\ F(t) + \overline{F(t)} = w(t), t \in L, \\ \left[F(z) + \overline{F(z)}\right]_{\Gamma_k} = w_k, k = 1, 2. \end{cases} \tag{22}$$

If the displacement's cyclic increment g_1 and the external resultant principal vectors F_2 are given, we get the boundary conditions for the second formulation as

$$(\mathbf{MDPP2})\quad \begin{cases} \kappa\phi_0(t) - t\overline{\phi_0'(t)} - \overline{\psi_0(t)} = 2\mu g^*(t) + i\beta_j t + c_j, t \in L_j, \\ \left[\kappa\phi_0(z) - z\overline{\phi_0'(z)} - \overline{\psi_0(z)}\right]_{\Gamma_1} = 2\mu g_1, \\ \left[\phi_0(z) + z\overline{\phi_0'(z)} + \overline{\psi_0(z)}\right]_{\Gamma_2} = iF_2, \\ \mathrm{Re}\int_{L_j} \psi_0(t)dt = M_j, j = 0, 1, \cdots, m-1, \\ F(t) + \overline{F(t)} = w(t), t \in L, \\ \left[F(z) + \overline{F(z)}\right]_{\Gamma_k} = w_k, k = 1, 2. \end{cases} \tag{23}$$

If the displacement cyclic increments g_1 and g_2 are given, we get the boundary conditions for the third formulation in the form

$$(\mathbf{MDPP3})\quad \begin{cases} \kappa\phi_0(t) - t\overline{\phi_0'(t)} - \overline{\psi_0(t)} = 2\mu g^*(t) + i\beta_j t + c_j, t \in L_j, \\ \left[\kappa\phi_0(z) - z\overline{\phi_0'(z)} - \overline{\psi_0(z)}\right]_{\Gamma_k} = 2\mu g_k, k = 1, 2, \\ \mathrm{Re}\int_{L_j} \psi_0(t)dt = M_j, j = 0, 1, \cdots, m-1, \\ F(t) + \overline{F(t)} = w(t), t \in L, \\ \left[F(z) + \overline{F(z)}\right]_{\Gamma_k} = w_k, k = 1, 2. \end{cases} \tag{24}$$

2 Solution

Because the multi-valued parts of $\phi(z)$ and $\psi(z)$ have been separated out, for convenience, we can assume $X_{1j}+iX_{2j} = 0, j = 0, 1, \cdots, m-1$. At first, let $\beta_j = 0, j = 0, 1, \cdots, m-1$. We consider

$$\kappa\phi_{00}(t) - t\overline{\phi_{00}'(t)} - \overline{\psi_{00}(t)} = 2\mu g^*(t) + c_{j0}, t \in L_j. \tag{25}$$

This is the boundary condition of the second fundamental problem with relative displacements to translation. Then, the Sherman transform takes the form,

$$\phi_{00}(z) = \frac{1}{2\pi i}\int_L \omega_0(t)\left[\zeta(t-z) - \zeta(t)\right]dt + \sum_{j=0}^{m-1} b_j\zeta(z-z_j) + Az, \tag{26}$$

$$\psi_{00}(z) = \frac{1}{2\pi i}\int_L \left\{\left[(\overline{\omega_0(t)}dt + \omega_0(t)d\bar{t}\right]\left[\zeta(t-z)-\zeta(t)\right]\right\} - \frac{1}{2\pi i}\int_L \omega_0(t)\left[\bar{t}\wp(t-z)-\rho_1(t-z)\right]dt + \sum_{j=0}^{m-1} b_j\left[\zeta(z-z_j)+\rho_1(z-z_j)\right] + Bz, \quad (27)$$

$$b_j = \frac{1}{2\pi i}\int_{L_j}\left[\omega_0(t)d\bar{t} - \overline{\omega_0(t)}dt\right], j = 0, 1, \cdots, m-1. \quad (28)$$

If $\wp(z)$ is the Weierstrass elliptic $\wp$ function, see [1],

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n}{}' \left\{\frac{1}{(z-\Omega_{mn})^2} - \frac{1}{\Omega_{mn}^2}\right\},$$

which has the properties

$$\wp(z) = \wp(-z),$$

$$\wp(z) = -\zeta'(z),$$

then $\rho_1(z)$ is a meromorphic function with the properties, see [2],

$$\rho_1(z+2\omega_k) - \rho_1(z) = 2\overline{\omega_k}\wp(z) + 2r_k, k = 1, 2. \quad (29)$$

r_1 and r_2 are known constants satisfying

$$r_2\omega_1 - r_1\omega_2 = \eta_1\overline{\omega_2} - \eta_2\overline{\omega_1} = -\frac{\pi i}{2}\delta_2. \quad (30)$$

Then, considering the boundary value problem for fixed k, $k = 1, \cdots, m-1$, respectively,

$$\kappa\phi_k(t) - t\overline{\phi_k'(t)} - \overline{\psi_k(t)} = i\delta_{kj}t + c_{jk}, t \in L_j, j = 0, 1, \cdots, m-1, \quad (31)$$

where δ_{kj} is the Kronecker symbol,

$$\delta_{kj} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

The representations used for the functions $\phi_k(z)$ and $\psi_k(z)$ in the present case are

$$\phi_k(z) = \frac{1}{2\pi i}\int_L \omega_k(t)\left[\zeta(t-z)-\zeta(t)\right]dt + \sum_{j=0}^{m-1} b_j\zeta(z-z_j), \quad (32)$$

$$\begin{aligned}\psi_k(z) = & \frac{1}{2\pi i}\int_L \left\{\left[(\overline{\omega_k(t)}dt + \omega_k(t)d\overline{t}\right]\left[\zeta(t-z)-\zeta(t)\right]\right\} \\ & -\frac{1}{2\pi i}\int_L \omega_k(t)\left[\overline{t}\wp(t-z)-\rho_1(t-z)\right]dt \\ & +\sum_{j=0}^{m-1} b_j\left[\zeta(z-z_j)+\rho_1(z-z_j)\right]. \end{aligned} \tag{33}$$

We know that if (26) and (27) are solution of equation (25), (32) and (33) are solution of equation (31), then

$$\begin{cases} \phi_0(z) = \phi_{00}(z) + \sum\limits_{k=1}^{m-1}\beta_k\phi_k(z), \\ \psi_0(z) = \psi_{00}(z) + \sum\limits_{k=1}^{m-1}\beta_k\psi_k(z), \\ C_j = C_{j0} + \sum\limits_{k=1}^{m-1} C_{jk}, j = 1,\cdots,m-1, \end{cases} \tag{34}$$

will be a solution of equation (15).

3 Example

As an example, the solution for (**MDPP2**) will be given.

From (1), (34) we have

$$\left\{\kappa\phi_{00}(z) - z\overline{\phi'_{00}} - \overline{\psi_{00}(z)} + \sum_{k=1}^{m-1}\beta_k\left[\kappa\phi_k(z) - z\overline{\phi'_k} - \overline{\psi_k(z)}\right]\right\}_{\Gamma_1} = 2\mu g_1, \tag{35}$$

$$\left\{\phi_{00}(z) + z\overline{\phi'_{00}} + \overline{\psi_{00}(z)} + \sum_{k=1}^{m-1}\beta_k\left[\phi_k(z) + z\overline{\phi'_k} + \overline{\psi_k(z)}\right]\right\}_{\Gamma_2} = iF_2, \tag{36}$$

Substituting (26), (27), (32) and (33) into equation (35) and (36), we get

$$\begin{cases} (\kappa A - \overline{A})\omega_1 - \overline{B}\overline{\omega_1} = \delta_1 + \mu g_1, \\ (A + \overline{A})\omega_2 + \overline{B}\overline{\omega_2} = \delta_2 + \frac{1}{2}F_2 i, \end{cases} \tag{37}$$

where

$$\delta_1 = \kappa b_0\eta_1 + \overline{a_0\eta_1} - \overline{b_0 r_1} + \sum_{k=1}^{m-1}\beta_k(\kappa b_k\eta_1 + \overline{a_k\eta_1} - \overline{b_k r_1}),$$

$$\delta_2 = b_0\eta_2 - \overline{a_0\eta_2} + b_0\overline{r_2} + \sum_{k=1}^{m-1}\beta_k(b_k\eta_2 - \overline{a_k\eta_2} + b_k\overline{r_2}),$$

$$a_0 = \frac{1}{\pi i}\int_L \overline{\omega_0(t)}dt, \quad a_k = \frac{1}{\pi i}\int_L \overline{\omega_k(t)}dt,$$

$$b_0 = \frac{1}{2\pi i}\int_L \left[\omega_0(t)d\bar{t} - \overline{\omega_0(t)}dt\right],$$

$$b_k = \frac{1}{2\pi i}\int_L \left[\omega_k(t)d\bar{t} - \overline{\omega_k(t)}dt\right].$$

Taking the complex conjugate of the system of equations (37), we obtain a system of equations with unknown A, $\overline{A}$, B and $\overline{B}$,

$$\begin{cases} \kappa\omega_1 A - \omega_1\overline{A} - \overline{\omega_1}\overline{B} = \delta_1 + \mu g_1, \\ \omega_2 A + \omega_2\overline{A} + \overline{\omega_2}\overline{B} = \delta_2 + \frac{1}{2}F_2 i, \\ -\overline{\omega_1}A + \kappa\overline{\omega_1}\overline{A} - \omega_1 B = \overline{\delta_1} + \mu\overline{g_1}, \\ \overline{\omega_2}A + \overline{\omega_2}\overline{A} + \omega_2 B = \overline{\delta_2} - \frac{1}{2}\overline{F_2}i. \end{cases} \tag{38}$$

The determinant of the system of equations (38) is

$$\begin{vmatrix} \kappa\omega_1 & -\omega_1 & 0 & -\overline{\omega_1} \\ \omega_2 & \omega_2 & 0 & \overline{\omega_2} \\ -\overline{\omega_1} & \kappa\overline{\omega_1} & -\omega_1 & 0 \\ \overline{\omega_2} & \overline{\omega_2} & \omega_2 & 0 \end{vmatrix} = -i\kappa S \text{ Re } (\omega_1\overline{\omega_2}) \neq 0, \tag{39}$$

where

$$S = 2i(\omega_1\overline{\omega_2} - \omega_2\overline{\omega_1}) \tag{40}$$

is exactly the area of the fundamental periodic-parallelogram P_{00} (on account of $Im(\omega_1) = 0$).

Hence, we can obtain A and B uniquely,

$$A = \frac{i\omega_1\overline{\omega_2}^2(\delta_1 + \mu g_1) - \kappa\omega_2\overline{\omega_1}^2(i\delta_2 - \frac{1}{2}F_2)}{\kappa S \text{ Re } (\omega_1\overline{\omega_2})}, \tag{41}$$

$$B = \frac{\kappa\omega_1^2\overline{\omega_2}(\frac{1}{2}\overline{F_2} + i\overline{\delta_2}) - i\omega_2^2\overline{\omega_1}(\overline{\delta_1} + \mu\overline{g_1})}{\kappa S \text{ Re } (\omega_1\overline{\omega_2})}. \tag{42}$$

Letting $z \to t_0 \in L$ and substituting (26) and (27) into equation (25), by employing the modified Plemelj formulae one obtains

$$\kappa\omega_0(t_0) + \frac{\kappa}{2\pi i}\int_L \omega_0(t)d\left[\log\frac{\sigma(t-t_0)\overline{\sigma(t)}}{\overline{\sigma(t-t_0)}\sigma(t)}\right]$$

$$+\frac{1}{2\pi i}\int_L \overline{\omega_0(t)}d\left[\overline{\zeta(t-t_0)} - (t-t_0)\overline{\zeta(t-t_0)}\right] \tag{43}$$

$$+M_1[\omega_0(t), t_0] = N_1(t_0), \tag{44}$$

where

$$M_1[\omega_0(t),t_0]=\frac{1}{2\pi i}\int_L\overline{\omega_0(t)}\left[t\overline{\wp(t-t_0)}-\overline{\rho(t-t_0)}\right]d\bar{t}$$
$$+\sum_{j=0}^{m-1}b_j\Big[2\text{ Re }\overline{\zeta(t_0-z_j)}+\overline{\rho_1(t_0-z_j)}-t_0\overline{\wp(t_0-z_j)}$$
$$+(\kappa-1)\zeta(t_0-z_j)\Big]$$
$$+\int_{L_j}\omega_0(t)ds+\kappa At_0-\overline{A}t_0+\overline{Bt_0},t_0\in L,$$

$$N_1(t_0)=2\mu\left[g(t_0)+e_3\nu t_0\right],$$
$$\zeta_0'(z)=-\rho(z),\zeta_0(0)=0,$$
$$g(t_0)=g_j(t_0),t_0\in L_j.$$

Substituting formulae (32) and (33) into equation (31) we get

$$\kappa\omega_k(t_0)+\frac{\kappa}{2\pi i}\int_L\omega_k(t)d\left[\log\frac{\sigma(t-t_0)\overline{\sigma(t)}}{\overline{\sigma(t-t_0)}\sigma(t)}\right]$$
$$+\frac{1}{2\pi i}\int_L\overline{\omega_0(t)}d\left[\overline{\zeta(t-t_0)}-(t-t_0)\overline{\zeta(t-t_0)}\right]\tag{45}$$
$$+M_2[\omega_k(t),t_0]=N_2(t_0),\tag{46}$$

where

$$M_2[\omega_k(t),t_0]=\frac{1}{2\pi i}\int_L\overline{\omega_k(t)}\left[t\overline{\wp(t-t_0)}-\overline{\rho(t-t_0)}\right]d\bar{t}$$
$$+\sum_{j=0}^{m-1}b_j\Big[2\text{ Re }\overline{\zeta(t_0-z_j)}+\overline{\rho_1(t_0-z_j)}-t_0\overline{\wp(t_0-z_j)}$$
$$+(\kappa-1)\zeta(t_0-z_j)\Big]$$
$$+\int_{L_j}\omega_k(t)ds+\kappa At_0-\overline{A}t_0+\overline{Bt_0},t_0\in L,$$

$$N_2(t_0)=i\delta_{kj}t_0.$$

After choosing $c_{00}=0$ and

$$c_{j0}=-\int_{L_j}\omega_0(t)ds,j=1,\cdots,m-1,\tag{47}$$

equation (43) has a unique solution $\omega_0^0(t)$. Then by the formulae (26), (27) and (47), one gets

$$\begin{cases} \phi_{00}(z) = \phi_{00}^0(z) + Az, \\ \psi_{00}(z) = \psi_{00}^0(z) + Bz, \\ c_{j0} = c_{j0}^0, \end{cases} \tag{48}$$

where

$$\phi_{00}^0(z) = \frac{1}{2\pi i}\int_L \omega_0^0(t)\left[\zeta(t-z) - \zeta(t)\right]dt + \sum_{j=0}^{m-1} b_j^0 \zeta(z - z_j), \tag{49}$$

$$\begin{aligned} \psi_{00}(z) \;=\; & \frac{1}{2\pi i}\int_L \left\{\left[(\overline{\omega_0^0(t)}dt + \omega_0^0(t)d\bar{t}\right]\left[\zeta(t-z) - \zeta(t)\right]\right\} \\ & -\frac{1}{2\pi i}\int_L \omega_0^0(t)\left[\bar{t}\wp(t-z) - \rho_1(t-z)\right]dt \\ & + \sum_{j=0}^{m-1} b_j^0\left[\zeta(z-z_j) + \rho_1(z-z_j)\right], \end{aligned} \tag{50}$$

$$b_j^0 = \frac{1}{2\pi i}\int_{L_j}\left[\omega_0^0(t)d\bar{t} - \overline{\omega_0^0(t)}dt\right], j = 0, 1, \cdots, m-1. \tag{51}$$

Similarly, after choosing $c_{0k} = 0$ and

$$c_{jk} = -\int_{L_j} \omega_k(t)ds, j = 1, \cdots, m-1, \tag{52}$$

equation (45) has a unique solution $\omega_k^0(t)$.

By formulae (32), (33) and (52), one has

$$\phi_k(z) = \phi_k^0(z), \quad \psi_k(z) = \psi_k^0(z), \quad c_{jk} = c_{jk}^0. \tag{53}$$

Substituting (34) into (21), taking $X_{1j} = X_{2j} = 0$ into account, one has immediately

$$\mathrm{Re}\int_{L_j}\left[\psi_{00}(t) + \sum_{k=1}^{m-1}\beta_k\psi_k(t)dt\right] = M_j, j = 1, \cdots, m-1, \tag{54}$$

or rewritten as

$$\sum_{k=1}^{m-1} A_{jk}\beta_k = M_j - B_j, j = 1, \cdots, m-1, \tag{55}$$

where

$$A_{jk} = \text{ Re } \int_{L_j} \psi_k(t)dt, \ j,k = 1, \cdots, m-1, \tag{56}$$

$$B_j = \text{ Re } \int_{L_j} \psi_{00}(t)dt, \ j = 1, \cdots, m-1. \tag{57}$$

Because

$$\int_{L_j} Btdt = 0, j = 1, \cdots, m-1, \tag{58}$$

then, all A_{jk}, B_j are known constants. Then (55) will be a system of linear algebraic equations with unknowns β_k. Now, we prove that (55) will be uniquely solvable. In fact, we only need to prove that the matrix (A_{jk}) is nonsingular. For this purpose, we consider the homogeneous condition: $g_j(t) = 0$, $X_{1j}(t) + iX_{2j}(t) = 0$, $M_j = 0$, $\ j = 0, 1, \cdots, m-1$. In this case, the real displacement is $i\alpha_j t + c'_j$. As in [9], consider the integral

$$J = \int_L (X_{1n}u + X_{2n}v)\, ds, \tag{59}$$

where

$$\begin{cases} X_{1n} = \sigma_1 cos(n, x_1) + \sigma_{12} cos(n, x_2), \\ X_{2n} = \sigma_{21} cos(n, x_1) + \sigma_2 cos(n, x_2). \end{cases} \tag{60}$$

By Green's theorem, we know the classical formula, see [9],

$$J = \iint\limits_{S_0} \left[\lambda(e_1 + e_2)^2 + 2\mu(e_1^2 + 2e_{12}^2 + e_2^2)\right] dx_1 dx_2. \tag{61}$$

In our case,

$$\begin{aligned} J &= \int_L (X_{1n}u + X_{2n}v)\, ds \\ &= \sum_{j=0}^{m-1} \int_{L_j} \left\{ \left[-\alpha_j x_2 + \text{ Re } (c'_j)\right] X_{1n} + \left[\alpha_j x_1 + Im(c'_j)\right] X_{2n} \right\} ds \\ &= \sum_{j=0}^{m-1} \Bigg[-\alpha_j \int_{L_j} (x_1 X_{2n} - x_2 X_{1n}) ds \\ &\qquad + \text{ Re } (c'_j) \int_{L_j} X_{1n} ds + Im(c'_j) \int_{L_j} X_{2n} ds \Bigg] \\ &= \sum_{j=0}^{m-1} \left[-\alpha_j M_j + \text{ Re } (c'_j) X_{1j} + Im(c'_j) X_{2j} \right]. \end{aligned} \tag{62}$$

Taking the above homogeneous conditions we have

$$J = 0. \tag{63}$$

Because the integrand on the right-hand side of (61) is a positive definite quadratic form, we have immediately,

$$e_1 = e_2 = e_{12} = 0. \tag{64}$$

Thus, there is only a rigid body displacement now, due to the fact that we have assumed $\alpha_0 = 0$. So there is no rotation any more. Therefore, all $\alpha_j = 0,\ j = 0, 1, \cdots, m-1$, furthermore, all $\beta_j = 2\mu\alpha_j = 0,\ j = 0, 1, \cdots, m-1$. This means the matrix (A_{jk}) is nonsingular, so we can obtain a unique solution β_j $(j = 0, 1, \cdots, m-1)$ from the system of equations (55).

By representing $F(z)$ as

$$F(z) = \frac{1}{2\pi i}\int_L i\Delta(t)\zeta(t-z)dt + Ez. \tag{65}$$

we may solve the last two boundary value problems of (**MDPP2**) similarly.

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Exact solution of the R–linear problem for a disk in a class of doubly periodic functions

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Abstract

The **R**–linear conjugation problem for a disk in the class of doubly periodic functions, i.e., the **R**–linear problem on the torus, is solved in the form of series by Eisenstein's functions. The result is applied to the calculation of the effective conductivity of composites with circular inclusions.

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Keywords: R-linear problem, effective conductivity, functional equation

1 Introduction

Let D_k be mutually disjoint simply connected domains in the complex plane $\mathbf{C}$ bounded by smooth curves ∂D_k $(k = 1, 2, \ldots, n)$, D be the complement of all the closures of ∂D_k with respect to the extended complex plane $\mathbf{C} \cup \{\infty\}$. Let ∂D_k be oriented in counter clockwise direction. Let $a(t)$, $b(t)$ and $c(t)$ be given Hölder continuous functions on $\partial D = -\cup_{k=1}^{n} \partial D_k$; $a(t) \neq 0$.

R–*linear conjugation problem:* Find a function $\varphi(z)$ analytic in $D, D_1, \ldots, D_n$, continuous in the closures of the considered domains with the following conjugation condition

$$\varphi^+(t) = a(t)\varphi^-(t) + b(t)\overline{\varphi^-(t)} + c(t), \quad t \in \partial D. \tag{1}$$

In the case $b(t) = 0$ we arrive at the **C**–linear conjugation problem [9].

Noether's theory for problem (1) has been constructed by Mikhajlov [14] by reducing it to a singular integral equation. Litvinchuk & Spitkovskii [11] studied the problem (1) for a circle by its reduction to a two-dimensional **C**–linear problem with a certain matrix coefficient.

This paper is devoted to constructive solution to the problem (1). First, we point to the case $a(t) \equiv b(t)$ in which Mikhajlov [14] has solved this problem by reduction it to the Riemann–Hilbert problem.

Special cases of problem (1) when it is possible to construct its solution are selected in [15], for instance, the **R**–linear conjugation problem with constant coefficients and circular inclusions ∂D_k. In this case (1) can be written in the form

$$\varphi^-(t) = \varphi^+(t) - \rho_k\overline{\varphi^+(t)} + c(t), \quad t \in \partial D_k,\ k = 1, 2, \ldots, n, \tag{2}$$

where ρ_k are constants. Frequently (2) is written in the following form [15]

$$\psi^-(t) = \psi^+(t) + \rho_k \left(\overline{n(t)}^2\right) \overline{\psi^+(t)} + c'(t), \quad t \in \partial D_k, \tag{3}$$

where $n(t)$ is the unit normal vector to ∂D_k written in the form of complex value, $\psi(z) = \varphi'(z)$. The **R**–linear problem (2) with $-1 \leq \rho_k \leq 1$ corresponds to the problem of perfect contact for composite materials with inclusions occupying the domains D_k. It has been solved in [15] by the method of functional equations for circular inclusions D_k.

Recently, Craster & Obnosov in a series of brilliant papers [5]–[8] have solved the problem (3) for contours forming checkerboard periodic structures on the plane.

As it follows from the homogenization theory [1] in order to determine the effective properties of the composites one has to consider the so-called cell periodicity problem. In the case of the conductivity of two-dimensional materials we arrive at the **R**–linear problem (3) in a class of doubly periodic functions. There are two interpretations of the latter problem. First, one can consider it as an **R**–linear problem on a torus represented by a rectangle with glued opposite sides [9, 26]. On the other hand this problem can be stated as a problem for infinitely connected domains [9, 24]. In the present paper we apply the first approach. The homogeneous **R**–linear problem for a disk on torus has been solved by reducing it to a functional equation. Following [16] we use Eisenstein's series to construct exact solution of the functional equation. This solution yields an exact formula for the effective conductivity of the square array of disks.

2 R–linear problem on a torus

Let $\mathbf{M} \cong \mathbf{Z}^2$ be the set of complex numbers with integer real and imaginary parts. Consider a square lattice $\mathcal{Q}$ which is defined by two fundamental translation vectors expressed by the complex numbers 1 and i on the complex plane $\mathbf{C}$. Introduce the zero-th cell $Q_0 := \{z = t_1 + it_2 : -1/2 < t_j < 1/2 \ (j = 1, 2)\}$. The lattice $\mathcal{Q}$ consists of the cell $Q_m := \{z \in \mathbf{C} : z - m \in Q_0\}$, where $m \in \mathbf{M}$. Let the disk $D_1 = \{z \in \mathbf{C} : |z| < r\}$ lie in the cell Q_0, D be the complement of the closure of D_1 to Q_0. To find a function $\psi(z)$ analytic in D, D_1, continuous in the closure of the considered domains with the following conjugation condition

$$\psi^-(t) = \psi^+(t) + \rho \left(\frac{r}{t}\right)^2 \overline{\psi^+(t)}, \quad |t| = r, \tag{4}$$

where $\psi(z)$ is doubly periodic with respect to $\mathcal{Q}$

$$\psi(z + 1) = \psi(z) = \psi(z + i). \tag{5}$$

The equality (3) is reduced to (4) for $|t| = r$ and $c'(t) = 0$. The given constant ρ satisfies the inequality $-1 < \rho < 1$. The problem (4)–(5) can be considered as the homogeneous **R**–linear problem for the unit circle on the torus represented by the cell Q_0. It can be also considered as an **R**–linear problem for the infinitely connected domain bounded by $|t - m| = r$ ($m = m_1 + im_2 \in \mathbf{M}$, i.e., m_1 and m_2 are integers).

In mechanics, the problem (4)–(5) corresponds to a problem for a composite material, when the conductivity of the matrix is normalized by unity and $\lambda_1 = \frac{1+\rho}{1-\rho}$ is the respective conductivity of the inclusions.

3 Classical Eisenstein-Rayleigh sums and Eisenstein series for the square lattice

In the present section we introduce the fundamental constants and functions of the elliptic function theory following Weil [25] and Akhiezer [2].

The Eisenstein summation method is defined as follows

$$\sum_{m_1,m_2} = \lim_{N\to\infty} \sum_{m_2=-N}^{m_2=N} \left(\lim_{M\to\infty} \sum_{m_1=-M}^{m_1=M} \right). \tag{6}$$

Using this summation we introduce the conditionally convergent sum

$$S_2 := \sum_{m_1,m_2}{}' \, (m_1 + im_2)^{-2} = \sum_{m}{}' \, m^{-2}, \tag{7}$$

where m_1 and m_2 run over all integer numbers except the pair $m_1 = m_2 = 0$. It is known [18] that $S_2 = \pi$. Following Eisenstein and Rayleigh we introduce the absolutely convergent sums

$$S_n := \sum_{m}{}' \, m^{-n}, \; n = 3, 4, \dots \; . \tag{8}$$

It is known that $S_n = 0$ for odd n. For even n an efficient algorithm has been proposed in [17] to calculate (8).

The Eisenstein series are defined as follows

$$E_n(z) := \sum_{m} (z - m)^{-n} \, , \; n = 2, 3, \dots \; . \tag{9}$$

The Eisenstein summation method (6) is applied to $E_2(z)$. The series $E_n(z)$ for $n = 3, 4, \dots$ as a function in z converge absolutely and almost uniformly in the domain $\mathbf{C}\backslash\mathbf{M}$. Each of the functions (9) is doubly periodic and has a pole of order n at $z = 0$.

The Eisenstein series and the Weierstrass function $\wp(z)$ are related by the identities

$$E_2(z) = \wp(z) + \pi, \tag{10}$$

$$E_n(z) = \frac{(-1)^n}{(n-1)!}\frac{d^{n-2}\wp(z)}{dz^{n-2}}, \quad n = 3, 4, \ldots \tag{11}$$

The Eisenstein functions of the even order $E_{2n}(z)$ can be presented in the form of the series

$$E_{2n}(z) = \frac{1}{z^{2n}} + \sum_{k=0}^{\infty} \sigma_k^{(n)} z^{2(k-1)}, \tag{12}$$

where

$$\sigma_k^{(n)} = \frac{(2n+2k-3)!}{(2n-1)!(2k-2)!} S_{2(n+k-1)}. \tag{13}$$

4 Reduction of the R–linear problem to a functional equation

In the present section we reduce the **R**–linear problem (4)–(5) to a functional equation. At first, we introduce the operator

$$T_m\psi(z) := \left(\frac{r}{z-m}\right)^2 \left(\overline{\psi\left(\frac{r^2}{\overline{z-m}}\right)} - \overline{\psi(0)}\right), \tag{14}$$

where $m \in \mathbf{M}$.

Introduce the Banach space C_A of functions continuous in $|z| \le r$ and analytic in $|z| < r$ with the norm $||\psi(z)|| = \max_{|t|=r} |\psi(z)|$.

Theorem [20] *Let $\sum_m$ and $\sum'_m$ denote Eisenstein's summation, respectively with the term $m = 0$ and without it.*

(i) The series

$$\Psi_0(z) = \sum_m{}' \, T_m\psi(z)$$

converges absolutely and uniformly in the closure of the cell Q_0 for each $\psi \in C_A$ to a function analytic in Q_0 and continuous in its closure.

(ii) The series

$$\Psi(z) = \sum_m T_m\psi(z) \tag{15}$$

converges absolutely and uniformly in each compact subset of D to a function analytic in D continuous in its closure and doubly periodic with respect to the considered lattice.

(iii) The linear operator

$$T = \sum_m{}' \, T_m \tag{16}$$

is compact in C_A.

Using Eisenstein's summation one can rewrite (15) in the form

$$\Psi(z) = \sum_m \left(\frac{r}{z-m}\right)^2 \overline{\psi\left(\frac{r^2}{\overline{z-m}}\right)} - \overline{\psi(0)} r^2 E_2(z), \tag{17}$$

since the sum

$$\Psi_e(z) = \sum_m \left(\frac{r}{z-m}\right)^2 \overline{\psi\left(\frac{r^2}{\overline{z-m}}\right)} \tag{18}$$

is correctly defined by Eisenstein's summation. It is commutative with integrals, it can be differentiated term by term as an absolutely and uniformly convergent series (15). However, it is forbidden to change the order of summation in (18).

We present the unknown function $\psi(z)$ in the form of its Taylor expansion

$$\psi(z) = \sum_{k=0}^{\infty} \psi_k z^k. \tag{19}$$

Then

$$\overline{\psi\left(\frac{r^2}{\overline{z-m}}\right)} = \sum_{k=0}^{\infty} \overline{\psi_k} r^{2k} \frac{1}{(z-m)^k} \tag{20}$$

for each m. Substitution of (20) in (18) yields

$$\Psi_e(z) = \sum_{k=0}^{\infty} \overline{\psi_k} r^{2k} E_{k+2}(z). \tag{21}$$

Introduce the function

$$\Phi(z) = \begin{cases} \psi(z) - \rho \sum\limits_{k=0}^{\infty} \overline{\psi_k} r^{2k} (E_{k+2}(z) - z^{-k-2}), \ |z| \le r, \\ \\ \psi(z) - \rho \sum\limits_{k=0}^{\infty} \overline{\psi_k} r^{2k} E_{k+2}(z), \ z \in D, \end{cases} \tag{22}$$

analytic in D and in the disk $|z| < r$. Calculate the jump of $\Phi(z)$ across $|t| = r$

$$\Delta = \Phi^+(t) - \Phi^-(t) = \psi^+(t) - \psi^-(t) - \rho \left(\frac{r}{t}\right)^2 \overline{\psi^-(t)}. \qquad (23)$$

Using (4) one can see that $\Delta = 0$. Applying the principle of analytic continuation and Liouville's theorem we obtain that $\Phi(z)$ is a constant, say c. This complex constant corresponds to the vector of the external flux applied to the composite [19]. Then the definition of $\Phi(z)$ in $|z| \le r$ yields the following functional equation with respect to $\psi \in C_A$

$$\psi(z) = \rho \sum_{k=0}^{\infty} \overline{\psi_k} r^{2k} (E_{k+2}(z) - z^{-k-2}) + c, \ |z| \le r, \qquad (24)$$

which can be also written in the form

$$\psi(z) = \rho \sum_{m}{}' \left(\frac{r}{z-m}\right)^2 \overline{\psi\left(\frac{r^2}{\overline{z-m}}\right)} + c, \ |z| \le r. \qquad (25)$$

We have

$$\overline{\psi\left(\frac{r^2}{\overline{z-m}}\right)} = \overline{\psi(z^*-m)},$$

where $z^* = \frac{r^2}{\overline{(z-m)}} + m$ is the inversion with respect to the circle $|z - m| = r$. The inversion z^* transforms the disk $|z| \le r$ onto a closed disk D^* from $|z - m| < r$ when $m \neq 0$. The translation $z \mapsto z - m$ returns D^* to $|z| < r$. Therefore, the shift $\frac{r^2}{z-m}$ maps $|z| \le r$ onto a closed disk lying in $|z| < r$. Equations with shifts into the domain are called iterative functional equations [10].

Theorem [20] *Equation* (24) *with* $|\rho| < 1$ *has a unique solution in* C_A. *This solution can be found by the method of successive approximations converging in* C_A, *i.e., uniformly convergent in* $|z| \le r$.

5 Solution to the functional equation

In the previous section we have noted that it is possible to apply the method of successive approximations to the functional equation (24) and hence to construct an approximate solution in symbolic form as it was done in [4, 19, 23]. However, in this case of one inclusion in the periodicity cell it is possible also to write explicitly each term of

this approximation, i.e., it is possible to write the exact solution of the functional equation in form of a series with explicitly given terms. It is convenient to perform it using the functional equation (25).

We are looking for $\psi(z)$ in the form of the series

$$\psi(z) = c\sum_{k=1}^{\infty} \left(\rho r^2\right)^k \psi_k(z). \tag{26}$$

Then (25) yields the following recurrence relations

$$\psi_0(z) = 1,$$
$$\psi_k(z) = {\sum_{m}}' \frac{1}{(z-m)^2} \overline{\psi_{k-1}\left(\frac{r^2}{\overline{z-m}}\right)},\ k = 1, 2, \dots\ . \tag{27}$$

Applying (27) we obtain

$$\begin{aligned}\psi_k(z) = {\sum_{\nu_1}}' {\sum_{\nu_2}}' \cdots {\sum_{\nu_k}}' (\nu_1 - z)^{-2} \left(\overline{\nu_2} - \frac{r^2}{\nu_1 - z}\right)^{-2} \\ \times \left(\nu_3 - \frac{r^2}{\overline{\nu_2} - \frac{r^2}{\nu_1 - z}}\right)^{-2} \cdots \left(\overline{\nu_k} - \frac{r^2}{\nu_{k-1} - \frac{r^2}{\overline{\nu_{k-2}} - \frac{r^2}{\dots}}}\right)^{-2},\end{aligned} \tag{28}$$

where ν_l corresponds to m from (27). For definiteness (28) is written for even k.

We apply (9), (12) and (13) to the latest term from (28)

$${\sum_{\nu_k}}' \left(\overline{\nu_k} - \frac{r^2}{\nu_{k-1} - \frac{r^2}{\overline{\nu_{k-2}} - \frac{r^2}{\dots}}}\right)^{-2} = \sum_{n_1=1}^{\infty} \sigma_{n_1}^{(1)} \frac{r^{4(n_1-1)}}{\left(\nu_{k-1} - \frac{r^2}{\overline{\nu_{k-2}} - \frac{r^2}{\dots}}\right)^{2(n_1-1)}}. \tag{29}$$

Substitution of (29) in (28) yields

$$\begin{aligned}\psi_k(z) = \sum_{n_1=1}^{\infty} \sigma_{n_1}^{(1)} r^{4(n_1-1)} {\sum_{\nu_1,\nu_2,\dots,\nu_k}}' (\nu_1 - z)^{-2} \left(\overline{\nu_2} - \frac{r^2}{\nu_1 - z}\right)^{-2} \\ \times \left(\nu_3 - \frac{r^2}{\overline{\nu_2} - \frac{r^2}{\nu_1 - z}}\right)^{-2} \cdots \left(\nu_{k-1} - \frac{r^2}{\overline{\nu_{k-2}} - \frac{r^2}{\dots}}\right)^{-2n_1},\end{aligned} \tag{30}$$

We now apply (9), (12) and (13) to the latest term from (30)

$$\begin{aligned}&\sum_{\nu_{k-1}}{}' \left(\nu_{k-1}-\frac{r^2}{\overline{\nu_{k-2}}-\frac{r^2}{\nu_{k-3}-\frac{r^2}{\dots}}}\right)^{-2n_1}\\&=\sum_{n_2=1}^{\infty}\sigma_{n_2}^{(n_1)}\frac{r^{4(n_2-1)}}{\left(\nu_{k-1}-\frac{r^2}{\overline{\nu_{k-2}}-\frac{r^2}{\dots}}\right)^{2(n_2-1)}}.\end{aligned}\tag{31}$$

Then (30) becomes

$$\psi_k(z)=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sigma_{n_1}^{(1)}\sigma_{n_2}^{(n_1)}r^{4(n_1+n_2-1)}\dots$$

$$\times\sum_{\nu_1,\nu_2,\dots,\nu_{k-2}}{}'(\nu_1-z)^{-2}\left(\overline{\nu_2}-\frac{r^2}{\nu_1-z}\right)^{-2}\dots\left(\overline{\nu_{k-2}}-\frac{r^2}{\nu_{k-3}-\frac{r^2}{\dots}}\right)^{-2n_2}.\tag{32}$$

We again apply (9), (12) and (13) to (32) and so forth. At the end we obtain the desired formula

$$\begin{aligned}\psi(z)&=c\sum_{k=0}^{\infty}(\rho r^2)^k\sum_{n_1,n_2,\dots,n_k}\sigma_{n_1}^{(1)}\sigma_{n_2}^{(n_1)}\dots\sigma_{n_{k-1}}^{(n_{k-2})}\\&\quad\times E_{n_k}(z)r^{4(n_1+n_2+\dots+n_k-k)}.\end{aligned}\tag{33}$$

Here we have used (26).

The effective conductivity of the square array is determined by the following equality [19] (there is also a proof that $\psi(0)/c$ is real)

$$\widehat{\lambda}=1+2\rho\pi r^2\frac{\psi(0)}{c}.\tag{34}$$

Substitution of (26) and (33) to (34) yields the exact formula

$$\widehat{\lambda}=1+2\pi\sum_{k=0}^{\infty}\rho^{k+1}\sum_{n_1,n_2,\dots,n_k}\sigma_{n_1}^{(1)}\sigma_{n_2}^{(n_1)}\dots\sigma_1^{(n_k)}r^{4(n_1+n_2+\dots+n_k)-2(k-1)}.\tag{35}$$

6 Conclusion

The formula (35) can be considered as an expansion of $\widehat{\lambda}$ on the concentration of the inclusions πr^2 and the contrast parameter ρ. This

analiticity is consistent with the previous general result of Bergman [3]. The formula (35) includes all known formulas for $\widehat{\lambda}$ approximated by πr^2 and ρ. However, in the case when $r \to 1/2$ and $\rho \to 1$ the series (35) diverges to $+\infty$. Asymptotic formulas for $\widehat{\lambda}$ in this case were obtained by McPhedran et al. [12] (see also papers cited therein). Direct application of (35) to this limit case is doubtful. One can find some notes on application of the functional equations to this case in [21].

It could be interesting to estimate $\psi(0)$ without direct solution to the problem (4) or to the functional equation (24) in the limit case. The general theory of bounds for $\widehat{\lambda}$ is presented by Milton [13] without an address to the **R**–linear problem.

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Holomorphic curves and Riemann-Hilbert problems in loop spaces

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Abstract

Geometric and topological aspects of holomorphic curves in loop spaces are discussed with a special emphasis on their relations to boundary value problems of Riemann-Hilbert type. A general formulation of Riemann-Hilbert problems in the setting of almost complex manifolds is presented and illustrated by considering some visual examples. In particular, analytic discs in the loop space of a three-dimensional Euclidean space are characterized in terms of a certain nonlinear partial differential equation. As an application it is shown that the Hopf fibration can be considered as solution to a Riemann-Hilbert problem in the so-called Brylinski loop space of a standard three-dimensional sphere.
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Key words: almost complex manifold, holomorphic curve, Riemann-Hilbert problem, loop space, Weisbach theorem, Hopf fibration, Villarceau circle

1 Introduction

A general concept of holomorphic curves in an almost complex manifold was suggested by M. Gromov [6]. As is well known, some loop spaces have natural almost complex structures (see, e.g., [3], [13], [15]), so one can define holomorphic curves in such loop spaces. Since holomorphic curves in complex vector spaces are closely related to boundary value problems for holomorphic functions it is natural to investigate those relations in the context of loop spaces. In the present paper we take an initial step in this direction.

We begin with describing a general setting for holomorphic curves and Riemann-Hilbert problems in the context of almost complex manifolds following [6], [7], [12], [20]. The general scheme is then applied to holomorphic functions with values in the space of loops immersed in a three-dimensional Riemannian manifold (3-fold). For brevity, in this context we speak of *loopy holomorphic curves* and *loopy Riemann-Hilbert problems*.

It turns out that such functions and their images are naturally connected with several classical geometric and analytic concepts and we study some aspects of those connections. In particular, a typical geometric picture of a loopy analytic disc is given by a solid torus foliated by closed curves satisfying a certain partial differential equation. Thus one can consider various geometric problems concerned with foliated solid tori which are closely related to solving loopy Riemann-Hilbert problems .

In order to make the exposition self-contained and precise (at least formally), in the second section we describe the basic construction of a natural almost complex structure on the space of immersed loops in a Riemannian 3-fold. This construction goes back to J.-L. Brylinski and L. Lempert, and we closely follow [3], [13].

In the third section we discuss geometric models of loopy analytic discs . Since the topic is sufficiently subtle and practically unexplored, our considerations are restricted to loops in a three-dimensional real affine space. In this case an explicit description of analytic discs is possible in terms of a certain partial differential equation suggested in [13] which we call the *Lempert equation*. A similar equation was obtained in [13] even in larger generality, but the Euclidean setting accepted in this paper enables us to give a natural representation of solutions to the Lempert equation in terms of conformal mappings (Theorem 2).

Our approach relies on *Weisbach's theorem*, sometimes called the *fundamental theorem of axonometry* [5], so we briefly recall it in a separate section and then explain its relation to Theorem 2. It may be added that Julius Weisbach was one of the most famous scientists at the Bergakademie Freiberg in the XIXth century (for details on Weisbach's life and his contributions to various fields see [16]), and it seems remarkable that the connection between Lempert's equation and Weisbach's theorem was established during a visit of the first author at the Bergakademie.

In conclusion we present an interpretation of the Hopf fibration as geometric picture of a holomorphic curve in the loop space of a three-dimensional sphere (Theorem 3). This interpretation also provides some nontrivial example of solutions to loopy Riemann-Hilbert problems and suggests further developments.

2 Riemann-Hilbert problems in almost complex setting

Nowadays there exist a number of commonly used concepts of Riemann-Hilbert problems . In this paper we only deal with Riemann-Hilbert problems considered as *boundary value problems for holomorphic functions*. A general formulation of a Riemann-Hilbert problem of such type was given in [7]. Here we elaborate upon the definition from [7] so that it becomes applicable to holomorphic functions of one complex variable with values in loop spaces of certain types.

To begin with, we present a version of the general definition from [7] which is appropriate for considering holomorphic functions of one complex variable with values in loop spaces of certain types. Recall that an almost complex structure J on a smooth manifold M is defined as a smooth family of linear operators $J_p = J(p)$ in tangent spaces $T_pM, p \in M$, such that $J_p^2 = -I$ (here and in the sequel I always denotes the identity mapping of the corresponding space). In particular, each complex manifold (for example, $\mathbb{C}^n$ or $\mathbb{CP}^n$) has a canonical complex structure defined by the operator of multiplication by $\imath$ in each tangent space. The concept of holomorphic mapping between complex manifolds can be generalized in the context of almost complex manifolds as follows.

Consider two almost complex manifolds (M, J) and (N, J'). A differentiable mapping $F : M \to N$ is called *holomorphic* if its differential

dF intertwines the given almost complex structures, namely:

$$dF(p)J_p = J'_{F(p)}dF(p), \tag{1}$$

for each $p \in M$. Sometimes such mappings are called *pseudo-holomorphic* (cf. [6]) but we prefer to omit the prefix "pseudo" since this cannot lead to a misunderstanding in the sequel. As is well known, the local description of such mappings is closely related to the *Bers-Vekua equation* and *generalized analytic functions* [18], [6].

It is easy to verify that, for finite-dimensional complex manifolds, the above definition gives the usual concept of holomorphic mapping. In particular, taking a domain in the complex plane endowed with the canonical complex structure we get a concept of holomorphic function of one complex variable with values in an almost complex manifold N. If M or/and N are infinite-dimensional complex manifolds modelled on complex Banach spaces, proving equivalence of the two definitions of holomorphic map requires more care but we need not discuss here those nuances.

If M is a one-dimensional complex manifold (Riemann surface) then the image of a holomorphic mapping $M \to N$ is called a *holomorphic curve* in N. In particular, if M is a simply connected domain in $\mathbb{C}$ such an image is called an *analytic disc*. If $M = \mathbb{CP}$ is the Riemann sphere then its holomorphic images are called *holomorphic spheres*. Obviously, a holomorphic sphere is a union of two analytic discs glued along their boundaries. In the last section we present an example of such a situation in the loop space of a 3-sphere.

In order to formulate the Riemann-Hilbert problem in an almost complex setting, suppose moreover that M is decomposed into two (open) parts M_+, M_- by a smooth divisor (hypersurface) Γ. We introduce a function spaces as follows. For an open subset $U \subset M$, let $A(U, N)$ denote the set of all continuous mappings from $\overline{U}$ into N which are holomorphic in U. Fix finally a continuous mapping (current) Φ on Γ with values in a subgroup G of the infinite-dimensional Lie-Frechet group Diff N consisting of smooth diffeomorphisms of N.

The Riemann-Hilbert problem defined by the quintuple (M, N, Γ, G, Φ) is formulated as the problem of describing the totality of pairs $(X_+, X_-) \in A((M_+, N) \times A(X_-, N))$ satisfying the following condition on the divisor Γ:

$$X_+(p) = \Phi(p)(X_-(p)),\ p \in \Gamma, \tag{2}$$

where $\Phi(p)$ acts on $X_-(p)$ as an element of DiffN. Notice that by taking $M = \mathbb{C}, \Gamma = \{|z| = 1\}, N = \mathbb{C}^n, G = GL(n, \mathbb{C})$, and some $(n \times n)$-matrix-function on Γ in the role of Φ, one obtains a classical version of the Riemann-Hilbert problem called the *problem of linear conjugation* (cf. [14], [20], [8]).

Thus we see that each pair of almost complex manifolds (M, N) yields a collection of analytic problems whose nature strongly depends on the geometry and topology of the manifolds and the group G considered. The classical theory of Riemann-Hilbert problems and singular integral equations appears as a particular case of this general scheme. In a short note like this one it makes no sense to make further comments on the general scheme, so we only wish to briefly outline some new aspects and phenomena appearing when N is taken to be an infinite dimensional almost complex manifold.

It should be noted that several classes of examples of such manifolds N are already known. Some of them arise from loop spaces of various kinds, the most famous examples being given by a group of loops in a compact Lie group [15] and the group Diff S^1/S^1 [15], [13]. Up to our knowledge, Riemann-Hilbert problems of the above type have not yet been discussed in the literature even for these concrete examples and we attempt to start filling this gap. In particular, we will try to reveal some peculiarities and perspectives arising in this context.

It appears convenient to use a little bit more geometric language. Namely, in complete analogy with the finite-dimensional setting, while dealing with Riemann-Hilbert problems in loop spaces one is inevitably led to considering images of holomorphic mappings of discs into a given loop space. Thus it becomes necessary to deal with analytic discs in loop spaces which could be called *loop-valued analytic discs*. For brevity, we call them *loopy analytic discs*.

Similar geometric objects in loop spaces of 3-folds were earlier considered by L. Lempert [13] and appeared to be related to interesting geometric constructions. In particular, some explicit and visual examples of loopy analytic discs can be found in [13].

Actually, in many cases solving Riemann-Hilbert problems can be reduced to constructing analytic discs with boundaries in a prescribed manifold (usually called a *target manifold* [20]). The latter problem is of geometric flavour and appears more flexible from the viewpoint of various generalizations, so in the sequel we concentrate on constructing loopy analytic discs.

It should be added that this language appeared useful in the theory of finite-dimensional non-linear Riemann-Hilbert problems [20]. In some sense, the present note may be also considered as an infinite-dimensional generalization of developments described in [12] which were concerned with analytic discs in finite-dimensional manifolds.

3 Immersed loops in Riemannian 3-folds

We explicate now the above general considerations in the context of certain loop spaces of Riemannian 3-folds. Namely, we give a precise definition of the loop space to be considered, together with a canonical almost complex structure on it. Given a Riemannian 3-fold M, the space we are going to deal with consists of the so-called *oriented immersed loops* in M [13]. This kind of loop space was carefully investigated by J.-L.Brylinski [3]. For this reason we call it the *Brylinski loop space* and denote it by BM in order to distinguish it from the usual space LM of loops considered with fixed parameterizations. Here and everywhere else we deal with smooth (meaning C^∞) loops.

Let M be an oriented Riemannian 3-fold. Denote by BM the set of equivalence classes of smooth immersions $f : S^1 \to M$ with respect to a natural equivalence relation: two loops $f_1, f_2 : S^1 \to M$ are equivalent if $f_1 = f_2 \circ \varphi$, where φ is an orientation preserving diffeomorphism of S^1. Elements of BM are sometimes called *immersed loops*. Notice that at each point $p \in M$ of an immersed loop $\gamma = [f]$ one has a well-defined tangent direction since we only permit orientable changes of variables. The unit vector in this direction is denoted by t_p.

The set BM can be endowed by a natural topology using the sections of the normal bundle in a standard way. Namely, fix an immersed loop $\gamma \in BM$ represented by $f : S^1 \to M$ and let N_f denote its normal bundle, i.e. the union of normal spaces to $\gamma = \operatorname{im} f$ in TM. All this makes sense since M has a Riemannian metric and one can, moreover, construct the (partially defined) exponential map $\exp : N_f \to M$. N_f inherits a Riemannian metric and connection from TM, so it makes sense to speak of C^k-norms of its smooth sections. Hence one can introduce the Whitney (or C^∞) topology on its smooth sections. The images of sufficiently small neighbourhoods of the origin under exp form by definition the basis of topology on BM.

As was shown in [3], BM endowed with this topology becomes a

Frechét manifold modelled on the space of smooth mappings from S^1 into $\mathbb{R}^2$. The tangent space $T_{[f]}BM$ can be naturally identified with $C^\infty(N_f)$ and its elements can be interpreted as normal vector fields along f.

This enables one to introduce an almost complex structure on BM in a pretty visual way. Namely, an endomorphism $J : T_{[f]}BM \to T_{[f]}BM$ is defined as follows. Let $v \in T_{[f]}BM$ and denote by $v(p)$ the corresponding normal vector at the point $p \in \mathrm{im}[f]$. Then we put $Jv = w$, where w is another normal vector field along f such that, for each point $p \in \mathrm{im}f$, the vector $w(p)$ is orthogonal to $v(p)$ and $t(p)$, has the same length as $v(p)$, and the triple $(v(p), w(p), t(p))$ is positively oriented (with respect to a given orientation of M). This obviously provides a completely rigorous definition of J. Actually, its action on a normal vector field along $[f]$ consists in rotation by $\frac{\pi}{2}$ in positive direction when looking from the endpoint of the tangent vector $t(p)$ (or equivalently the vector product $v(p) \times w(p)$ points in the direction of the tangent vector $t(p)$). It is pretty obvious that $J^2 = -I$ so that J defines an almost complex structure on BM which is called the *canonical almost complex structure*.

We now pass to studying analytic discs in the Brylinski loop space BM, where M is an oriented Riemannian 3-fold as above. An especially visual and practically important case is when the ambient manifold is a *three-dimensional real vector space with Euclidean metric* which we call *Euclidean 3-space* for short. Another visual case is when one takes the ambient manifold to be a three-dimensional sphere S^3 with the standard (round) metric induced from the ambient four-dimensional Euclidean space. Such a sphere will be called a *Euclidean 3-sphere* or *round 3-sphere*.

Our next aim is to obtain a more explicit description of loopy analytic discs in the Euclidean 3-space and the round 3-sphere. In fact, in both these cases it even appears possible to give a visual analytic interpretation for loopy analytic discs and relate them to certain nonlinear differential equations, which will be our main occupation in the rest of this paper.

4 Loopy analytic discs in $\mathbb{R}^3$

We recall now an analytic description of Euclidean loopy analytic discs obtained by L. Lempert [13]. Let us first describe the typical geometric

picture of a generic loopy analytic disc in any 3-fold M. Notice that geometrically it is just a 2-parametric family of loops in M with a parameter running on some disc which at the first step can be assumed sufficiently small. Then it is easy to imagine that generically loops in the family are non-singular and mutually disjoint. Thus their union should be homeomorphic to the direct product $D^2 \times S^1$, i.e., to a solid torus $\mathbb{T}$. Moreover, this solid torus comes together with a foliation by closed curves in which case we speak of a *foliated solid torus*. This is actually a well-known pattern in three-dimensional topology because such objects serve as building blocks for the so-called *Seifert fibrations* [17], [11].

A natural problem which immediately arises in this context, is to investigate if a given foliated solid torus can be represented as the image of an analytic disc. One important case where this can be done was indicated by L. Lempert [13]. Namely, in the case of the Euclidean 3-space, loopy analytic discs can be constructed from solutions to a rather simple nonlinear differential equation for functions of three real variables.

To this end L. Lempert associated a complex-valued function with a given foliated solid torus $\mathbb{T} \subset \mathbb{R}^3$ as follows. Intersect $\mathbb{T}$ with a hypersurface H which transversally meets each leaf in a single point. Then the intersection $H \cap \mathbb{T}$ is diffeomorphic to a disc $D \subset \mathbb{C}$ and can be identified with this disc, so its points can be simply considered as complex numbers. Notice that by our assumption, for each point $p = (x, y, z) \in \mathbb{T}$, there exists a unique leaf passing through p and by construction it intersects $H \cap \mathbb{T}$ in a single point which we denote by $u(x, y, z)$. This correspondence defines a function $u : \mathbb{T} \to \mathbb{C}$ and from general principles it follows that generically u is a differentiable complex-valued function on U. We call it a *generating function* of the foliated solid torus $\mathbb{T}$. Obviously, it depends on the choice of the transversal hypersurface H.

Among other results, L. Lempert proved in [13] that if a given foliated solid torus $\mathbb{T}$ arose as the image of a loopy analytic disc on the Euclidean 3-space, then H can be chosen in such way that the generating function u satisfies the following nonlinear partial differential equation,

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 0, \tag{3}$$

which we call the *Lempert equation*. Actually, in this case the intersection $H \cap \mathbb{T}$ can be obtained by taking values of loops at any fixed

point $t \in S^1$.

The above equation is quite interesting by itself and its analysis is crucial for constructing loopy analytic discs. Actually, a sort of converse is true: speaking informally, loopy analytic discs can be constructed from solutions to the Lempert equation. This result is especially important for us, so we formulate it as a theorem. The proof is omitted because it is actually just a little bit more precise version of a similar statement established in [13].

Theorem 1 ([13]) *If $\mathbb{T}$ is a foliated solid torus in the Euclidean 3-space and there exists a hypersurface H such that the generating function u is differentiable and satisfies equation* (3) *then the leaves of foliation can be reparameterized in such a way that $\mathbb{T}$ becomes the image of a loopy analytic disc.*

Thus for constructing loopy analytic discs and solving loopy Riemann-Hilbert problems it is crucial to have at hand an ample set of solutions to the Lempert equation. We proceed towards that aim by relating its solutions to conformal mappings in three dimensions. Our approach relies on a basic result of axonometry, proved by Julius Weisbach in 1844. In the next section we recall this theorem and explain how it can be applied to constructing loopy analytic discs.

5 Weisbach theorem and conformal lifts

Consider the three-dimensional Euclidean space $E = \mathbb{R}^3$ with canonical coordinates (x, y, t) and identify the hyperplane $L = \{t = 0\}$ with the standard complex plane $\mathbb{C}$ by putting $z = x + \imath y$. Under an *orthogonal frame* in E we understand any triple of radius-vectors (i.e., vectors with the initial point at the origin) which are mutually orthogonal and have the same length.

Let us say that three points of L are called *contactical* if they can be represented as the images of orthogonal projection of the ends of some orthogonal frame. The following classical result is easy to prove ([1], [5]) but quite useful (e.g., in technical drawing).

Weisbach theorem. *Three points $p, q, r \in L \cong \mathbb{C}$ are contactical if and only if*

$$p^2 + q^2 + r^2 = 0. \tag{4}$$

This theorem permits a visual geometric interpretation which makes its proof nearly evident (notice that the "only if" part can be verified by an elementary computation). In order to proof the "if" part, consider the triples of real and complex parts of p, q, r as vectors X, Y in $\mathbb{R}^3$. Then it is immediate that condition (4) implies that X and Y are orthogonal and have the same length $d = |X| = |Y|$. It is also easy to see that in this setting our task is just to find a third vector Z such that it is orthogonal to both X and Y and has the same length. Existence of such a vector is geometrically obvious but one can also write a useful explicit formula in terms of the vector product, namely:

$$Z = \pm\frac{X \times Y}{d}, \tag{5}$$

where the sign can be chosen according to a desired orientation of the triple (X, Y, Z). This completes the proof and reveals the geometric meaning of Weisbach theorem.

Moreover, such an interpretation of this result enables one to establish a very explicit relation between conformal mappings and solutions of the Lempert equation. Indeed, let us consider a germ of a conformal mapping $\Phi : E \to E$. Then by conformality condition its differential transforms each orthogonal frame into an orthogonal one. Consider now the composition $v = P \circ \Phi$, where P is the orthogonal projection on L. Since P is linear, using the chain rule one easily derives that v is a solution to Lempert equation. Thus generating functions for Euclidean loopy analytic discs can be obtained from germs of conformal mappings. This is the key result in the sequel and so we formulate it as a theorem.

Theorem 2 *Each germ h of a conformal map in $\mathbb{R}^3$ defines a germ of a solution to Lempert's equation* (3) *by taking its composition $P \circ h$ with the orthogonal projection P.*

Taking Theorem 1 into account, this result provides an ample source of loopy analytic discs . One may wonder about the converse, namely, if each germ u of a solution to Lempert's equation (3) can be obtained in such a way. If this is the case, i.e., if there exists a germ of a conformal mapping h such that $u = P \circ h$ then any such germ is called a *conformal lift* of u. Since by Liouville theorem the storage of conformal maps in a three-dimensional Euclidean space is rather poor, there is no hope that conformal lifts always exist. Therefore it is desirable to indicate some reasonable criterion or sufficient

conditions for the existence of conformal lifts. This problem can be successfully treated using Weisbach's theorem and our comments to its proof.

More precisely, let u be a solution to Lempert equation (3) in a domain U. Separating the real and imaginary parts of its gradient we get two gradient vector fields X, Y in U. It is straightforward to verify that in order to obtain a conformal lift of u it is sufficient to find a function germ ψ such that its gradient $Z = \operatorname{grad} \psi$ taken together with X, Y forms an orthogonal frame at each point $p \in U$. Notice that at each point the sought field Z can be expressed by the formula (5) with the sign chosen once and forever.

So the problem is reduced to showing that the vector field defined by this formula is actually a *gradient* vector field. To this end one can use the well known criterion in terms of vanishing of the curl (rotor). Using standard formulae of vector calculus one can now get an explicit expression for the curl of the vector field defined by formula (5) and write the conditions which guarantee vanishing of the rotor in question.

This gives a system of explicit partial differential equations which should be satisfied by u in order that it has a conformal lift. By an easy qualitative analysis of those equations one becomes convinced that conformal lifts do not always exist. Thus one should look for further methods of constructing solutions to Lempert equation, which is an interesting open problem.

However the above results already enable one to construct many examples of loopy analytic discs in $\mathbb{R}^3$. Some of them can be written explicitly so that one becomes able to visualize loopy analytic discs in a quite satisfactory way.

Another merit of the above connection between loopy analytic discs and conformal mappings is its conformal invariance, so applying the stereographic projection $S^3 \to \mathbb{R}^3$ one can obtain an analog of Lempert's equation for a round 3-sphere and construct examples of loopy analytic discs using Theorem 1. As an illustration, in conclusion we describe a curious visual example of such kind arising from the Hopf fibration.

6 Hopf fibration as a holomorphic curve

For our purposes it is appropriate to define the Hopf fibration in complex setting. Consider the unit sphere $S^3 \subset \mathbb{C}^2 \cong \mathbb{R}^4$ and the Riemann sphere $\mathbb{P} = \overline{\mathbb{C}} \cong S^2$. Then the Hopf fibration $H : S^3 \to S^2$ is defined by sending each point $(z_1, z_2) \in S^3$ onto the ratio of its coordinates interpreted as a point of $\mathbb{P}$, i.e. $H(z_1, z_2) = z_1/z_2$. It is evident that fibres of H are the *complex great circles*, i.e. intersections of complex lines in $\mathbb{C}^2$ with S^3, so one can consider its "inverse" as a map from S^2 into the space of smooth loops on S^3.

Let us endow S^3 with the standard Riemannian metric inherited from the ambient Euclidean space. The sphere S^3 endowed with this metric will be called the round 3-sphere and denoted by S^3_r. We can now consider the corresponding Brylinski loop space BS^3_r and get a map $H^{-1} : \mathbb{P} \to BS^3_r$. Thus it becomes possible to treat the latter map from the viewpoint developed in the previous sections. A straightforward calculation shows that its differential dH^{-1} intertwines the almost complex structures on $\mathbb{P}$ and BS^3_r and so it defines a holomorphic curve in BS^3_r. We omit this calculation because the same conclusion may be obtained in a much more direct and visual way by considering the examples of analytic discs in $B\mathbb{R}^3$ described below.

Correspondingly, the restriction of H^{-1} on any disc in $\mathbb{P}$ defines a loopy analytic disc in BS^3_r. In particular, taking the unit disc and its complement we get a solution to the loopy Riemann-Hilbert problem (2) with the constant coefficient whose value at each point $p \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is the identity mapping of BS^3_r. In terms of analytic discs, one can state that S^3_r is the union of images of two loopy analytic discs. Despite their simplicity, these observations seem instructive and we formulate them as a theorem.

Theorem 3 *The map H^{-1} defines a holomorphic embedding of the Riemann sphere into the space of oriented immersed loops on S^3_r, in other words, H^{-1} defines a holomorphic curve in BS^3_r. In particular, a round 3-sphere S^3_r can be represented as a union of two loopy analytic discs glued along their boundaries, i.e., S^3_r foliated by the complex great circles is a solution to a loopy Riemann-Hilbert problem in BS^3.*

Using the above observations and the stereographic projection $\Pi : S^3 \to \mathbb{R}^3$ one can get a nice geometric picture in $\mathbb{R}^3$, which, in particular, enables one to visualize certain analytic discs in $\mathbb{R}^3$. It

is well known that the image of the unit disc under $\Phi = \Pi \circ H^{-1}$ is a solid torus $\mathbb{T}$ bounded by a round torus (torus of revolution) $\mathbb{T} \cong T^2$ in $\mathbb{R}^3$ (see, e.g., Ch.10 in [1]). The same holds for any disc in $\mathbb{C} \subset \mathbb{P}$ centered at the origin.

It is also known (but probably not so "well-known") that the images of complex great circles under Π are genuine (metric) circles which have been discovered by I. Villarceau [19] (nowadays they are called *Villarceau circles* [1]). They can be defined as the intersections of a round torus T^2_r with the bitangent plane passing through the center of the torus T^2_r [1]. Thus the preimages $\Phi^{-1}(w)$ of points w from the unit disc are exactly the Villarceau circles.

On each round torus T^2_r, Villarceau circles come in two families each of which consists of nonintersecting circles. Two Villarceau circles belonging to the same family will be called *coherent*. Thus each of the two families of coherent Villarceau circles defines a foliation of a round torus. Any two circles in the same family on a given round torus are linked with the linking number 1 which corresponds to the well-known fact that the Hopf invariant of Hopf fibration is equal to one.

Consider now a *round solid torus* $\mathbb{T}_r$ defined as the closure of interior of a round torus T^2_r. Obviously, $\mathbb{T}_r$ is a union of the continual family of coaxial round tori lying inside $\mathbb{T}_r$ and the axial circle which is equal to the intersection of their interiors. One sees now that the family of coherent Villarceau circles of all those round tori can be chosen in such a way that, together with the axial circle, they give a foliation of $\mathbb{T}_r$ by loops (circles) which are mutually linked with the linking number 1. Let us call this picture a *Villarceau solid torus*.

Comparing these observations with the discussion in Section 3 we see that a Villarceau solid torus geometrically resembles the geometric picture of analytic disc described in the preceding section. In fact, Theorem 1 can be used to show that Villarceau foliations really define loopy analytic discs in $\mathbb{R}^3$.

To this end we simply indicate a generating function in the sense of Section 3 which is naturally associated with the above geometric picture. Consider the following complex-valued function of three complex variables

$$u(x, y, z) = \frac{x^2 + y^2 + (z - \imath)^2}{x + \imath y}$$

which is merely the composition $\Pi_2 \circ H \circ \Pi^{-1}$ written in Cartesian coordinates, where Π_2 denotes the stereographic projection $S^2 \to \mathbb{R}^2 \cong \mathbb{C}$.

Taking the above remarks into account we conclude that the preim-

age of each point $w \in \mathbb{C}$ is a Villarceau circle. Moreover, the full preimage of any circle centered at the origin is a round solid torus. Now it is easy to see that the function u is a generating function for the Villarceau solid tori described above. In order to see that it really makes each of those solid tori into a loopy analytic discs it remains to verify that it satisfies the Lempert equation, which is a matter of elementary check. By Theorem 2 this means that each of those Villarceau round tori gives a precise picture of a loopy analytic disc in $\mathbb{R}^3$ which we call *Villarceau toroid*. Thus we have established the following final result.

Proposition 1 *Each round solid torus in $\mathbb{R}^3$ foliated by Villarceau circles is the image of an analytic disc in $B\mathbb{R}^3$.*

Up to our mind, this beautiful geometric picture alone gives a sufficient justification for the setting and considerations presented above. Since the main aim of this paper was just to establish the setting and demonstrate its fertility we delay a more detailed discussion for the future and conclude by adding a few remarks related to the latter proposition.

First of all, since the loopy analytic discs in S^3_r defined by H^{-1} and Villarceau toroids are related by the stereographic projection which is a conformal map, from the chain rule and the above computation for the function u it follows that H^{-1} is indeed a holomorphic map into BS^3_r. This gives a simple proof of Theorem 3 avoiding explicit use of the round metric on S^3_r. It is also worthy of noting that Proposition 1 could be derived from Theorem 2 by showing that the function u has a conformal lift, which follows from its relation to stereographic mapping. In particular, one can verify the vanishing of the rotor of the normalized vector product from the formula (5).

Next, using "general principles" of nonlinear analysis it is possible to show that one can deform the Villarceau toroid in such a way that all leaves of the foliation remain closed and the generating function is again a solution to Lempert's equation. Such deformations can be described by explicit equations using methods of deformation theory. For us the main point is that they provide examples of loopy analytic discs different from Villarceau toroids.

Proposition 2 *There exist small perturbations of the Villarceau toroid which can be represented as the images of loopy analytic discs.*

With some additional effort this fact provides solutions to loopy Riemann-Hilbert problems with non-constant coefficients which are

sufficiently close to the identity. It would be very interesting to construct similar examples with coefficients not necessarily close to identity.

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Haseman boundary value problem for bianalytic functions with different shifts on the unit circumference*

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Abstract

In this article, the Haseman boundary value problem of bianalytic function with both different factors and different shifts on the unit circumference is investigated. The expression for the solution and the condition of solvability of the Haseman boundary value problem are obtained by reducing the problem to the equivalent Haseman boundary value problem for analytic functions.

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Keywords: bianalytic function, Hasemann boundary value problem, canonical function

1 Introduction

The concept of polyanalytic function is introduced in [1]. In recent years, there are many authors who have already investigated various kinds of boundary value problems (BVPs) for polyanalytic functions and their generalization to metaanalytic functions [2, 3, 4, 5, 6]. In general, the condition of solvability and the expression of solution for those BVPs are obtained by two methods, which are the canonical method [3] and the conversion method [2, 4, 5, 6] respectively. The canonical method is based on the classical idea on BVPs of analytic functions by so-called poly-Cauchy type integral operators and the conversion method is to transform directly BVPs into the equivalent systems of BVPs for analytic functions by the decomposition theorem of polyanalytic functions. The Haseman BVP for bianalytic function with matrix factor but the same shift have been studied in [4] by the conversion method. In [4] the condition of solvability and the expression of solution depend on the so-called canonical matrix. In this article, the Haseman BVP for bianalytic functions with both different factors and different shifts on the unit circumference are investigated, and the condition of solvability and the expression of the solution are obtained by reducing the problem to BVPs for analytic functions.

Let Π_n denote the class of all polynomials of degree n at most and regard $\Pi_n = \{0\}$ while $n < 0$. Suppose that L oriented counterclockwise, is a simple closed Lyapnuov curve which divides the complex plane into two domains: S^+ including the origin and S^- including infinity. In [4], we know that $S^n_\alpha = \{p_n(\omega_\alpha(z) - \omega_\alpha(0)), p_n \in \Pi_n\}$ is the set of solutions of the following BVP for analytic functions with the positive shift α [7, 8, 9]

$$\begin{cases} \omega^+(\alpha(t)) = \omega^-(t), \ t \in L, \\ \mathrm{Ord}(\omega, \infty) \le n, \end{cases} \tag{1}$$

where $\mathrm{Ord}(\omega, \infty)$ represents the order of ω at infinity, $\alpha(t), \alpha'(t) \in H(L)$, $\alpha'(t) \neq 0$, and

$$\omega_\alpha(z) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_L \frac{\mu(\alpha^{-1}(t))dt}{t-z}, & z \in S^+, \\ \dfrac{1}{2\pi i}\displaystyle\int_L \frac{\mu(t)dt}{t-z} + z, & z \in S^-, \end{cases} \tag{2}$$

with the unique solution $\mu(t)$ of the Fredholm integral equation

$$(\mathbf{K}\mu)(t) = \mu(t) + \frac{1}{2\pi i}\int_L \left[\frac{\alpha'(\tau)}{\alpha(\tau)-\alpha(t)} - \frac{1}{\tau - t}\right]\mu(\tau)d\tau = t,\ t \in L. \tag{3}$$

The result will be used later.

2 Haseman bvp for bianalytic functions

Suppose that $\partial D = \{t : |t| = 1\}$ is the unit circumference, oriented counter-clockwise, which divides the complex plane $\mathbb{C}$ into two domains: $0 \in D^+$ and $\infty \in D^-$. Now our problem is to find a bianalytic function $V(z)$ with a jump on ∂D, $V(z)$ and $\frac{\partial V(z)}{\partial \bar{z}}$ being continuous on $\overline{D^+}$ and $\overline{D^-}$, satisfying the following Haseman type boundary condition and growth condition

$$\begin{cases} V^+(\alpha_1(t)) = G_1(t)V^-(t) + g_1(t), & t \in \partial D, \\ \left[\dfrac{\partial V}{\partial \bar{z}}\right]^+(\alpha_2(t)) = G_2(t)\left[\dfrac{\partial V}{\partial \bar{z}}\right]^-(t) + g_2(t), & t \in \partial D, \\ \mathrm{Ord}(V, \infty) \le 0, \end{cases} \tag{4}$$

where $G_j(t)$, $g_j(t)$ $(j = 1, 2)$ satisfy the Hölder condition on ∂D, $G_j(t) \neq 0$ $(j = 1, 2)$ on ∂D, and $\alpha_1(t), \alpha_2(t)$ are two positive shifts as that in §1. This problem is called BSR_0 problem [4].

Let

$$\kappa_j = \frac{1}{2\pi}\big[\arg G_j(t)\big]_{\partial D},\ j = 1, 2. \tag{5}$$

Setting

$$V(z) = V_1(z) + \bar{z}V_2(z), \tag{6}$$

where $V_1(z), V_2(z)$ are analytic on $D^+ \cup D^-$ and continuous up to ∂D [3]. We know $\mathrm{Ord}(V_1, \infty) \le 0$ and $\mathrm{Ord}(V_2, \infty) \le -1$ by Corollary 2.1 in [3]. Putting (6) into (4), considering $|\alpha_1(t)| = 1$ and $|t| = 1$ on ∂D, then (4) is equivalent to

$$\begin{cases} \widetilde{V}_1^+(\alpha_1(t)) = \widetilde{G}_1(t)\widetilde{V}_1^-(t) + \widetilde{g}_1(t),\ t \in \partial D, \\ \mathrm{Ord}(\widetilde{V}_1, \infty) \le 1, \end{cases} \tag{7}$$

and

$$\begin{cases} V_2^+(\alpha_2(t)) = G_2(t)V_2^-(t) + g_2(t),\ t \in \partial D, \\ \mathrm{Ord}(V_2, \infty) \le -1, \end{cases} \tag{8}$$

where

$$\begin{cases} \widetilde{G}_1(t) &= \dfrac{G_1(t)\alpha_1(t)}{t} \in H(\partial D), \\ \widetilde{g}_1(t) &= g_1(t)\alpha_1(t) \in H(\partial D), \end{cases} \tag{9}$$

and $\widetilde{V}_1^{\pm}(t)$ is the boundary value of the function

$$\widetilde{V}_1(z) = zV_1(z) + V_2(z). \tag{10}$$

Obviously,

$$\begin{cases} \widetilde{G}_1(t) \neq 0,\ t \in \partial D, \\ \dfrac{1}{2\pi}\left[\arg \widetilde{G}_1(t)\right]_{\partial D} = \kappa_1. \end{cases} \tag{11}$$

From [7, 8], let

$$X_1(z) = \begin{cases} \exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_{\partial D} \frac{\rho_1(\alpha_1^{-1}(\tau))d\tau}{\tau - z}\right\}, & z \in D^+, \\ z^{-\kappa_1}\exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_{\partial D} \frac{\rho_1(\tau)d\tau}{\tau - z}\right\}, & z \in D^-, \end{cases} \tag{12}$$

where $\rho_1(t)$ is the unique solution of the following Fredholm integral equation

$$(\mathbf{K}\rho_1)(t) = \log\left\{t^{-\kappa_1}\widetilde{G}_1(t)\right\},\ t \in \partial D, \tag{13}$$

where the operator $\mathbf{K}$ is defined by (3). $X_1(z)$ is the so-called canonical function of the homogeneous Haseman BVP of (7). It possesses the following properties:

(P1) $X_1 \in A^+H(D^+)(A^-H(D^-))$, i.e., X_1 is analytic on $D^+(D^-)$ and $\in H$ on $\overline{D^+}(\overline{D^-})$,

(P2) $X_1^{\pm}(z) \neq 0$ for $z \in \mathbb{C}$,

(P3) $\widetilde{G}_1(t) = [X_1^+(\alpha_1(t))][X_1^-(t)]^{-1}$,

(P4) $\mathrm{Ord}(X_1, \infty) = -\kappa_1$, exactly, $\lim\limits_{z\to\infty} z^{\kappa_1}X_1(z) = 1$.

By Property 3 of the canonical function X_1, the boundary condition in (7) is

$$[X_1^+(\alpha_1(t))]^{-1}\widetilde{V}_1^+(\alpha_1(t)) = [X_1^-(t)]^{-1}\widetilde{V}_1^-(t) + [X_1^+(\alpha_1(t))]^{-1}\widetilde{g}_1(t). \tag{14}$$

Setting

$$\Psi_1(z)=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{\partial D}\frac{\chi_1(\alpha_1^{-1}(\tau))d\tau}{\tau-z}, & z\in D^+,\\ \dfrac{1}{2\pi i}\displaystyle\int_{\partial D}\frac{\chi_1(\tau)d\tau}{\tau-z}, & z\in D^-,\end{cases}\tag{15}$$

where $\chi_1(t)$ is the unique solution of the Fredholm integral equation

$$(\mathbf{K}\chi_1)(t)=[X_1^+(\alpha_1(t))]^{-1}\widetilde{g}_1(t),\ t\in\partial D,\tag{16}$$

by the Plemelj formula, we obtain

$$(\mathbf{K}\chi_1)(t)=\Psi_1^+(\alpha_1(t))-\Psi_1^-(t)=[X_1^+(\alpha_1(t))]^{-1}\widetilde{g}_1(t).\tag{17}$$

Hence, by (14) and (17),

$$[X_1^+(\alpha_1(t))]^{-1}\widetilde{V}_1^+(\alpha_1(t))-\Psi_1^+(\alpha_1(t))=[X_1^-(t)]^{-1}\widetilde{V}_1^-(t)-\Psi_1^-(t).\tag{18}$$

Thus, the problem (7) is solvable if and only if the conditions

$$\int_{\partial D}\tau^k\chi_1(\tau)d\tau=0,\ k=0,1,\cdots,-\kappa_1-3,\tag{19}$$

are satisfied, and by the result of (1) stated in Section 1 we get

$$[X_1(z)]^{-1}\widetilde{V}_1(z)-\Psi_1(z)=q_{\kappa_1+1}^{\alpha_1}(z),\tag{20}$$

i.e.,

$$zV_1(z)+V_2(z)=X_1(z)\left[\Psi_1(z)+q_{\kappa_1+1}^{\alpha_1}(z)\right],\tag{21}$$

where χ_1 is the unique solution of (16) and $q_{\kappa_1+1}^{\alpha_1}\in S_{\alpha_1}^{\kappa_1+1}$. If $-\kappa_1-3<0$ we agree that the solvability conditions (19) do not arise.

Remark 1 The conditions of solvability (19) may rewritten as

$$\int_{\partial D}h_{1k}(\tau)\widetilde{g}_1(\tau)d\tau=0,\ k=0,1,\cdots,-\kappa_1-3,$$

with

$$h_{1k}(\tau)=\frac{1}{X_1^+(\alpha(\tau))}\left[\tau^k+\int_{\partial D}R(\tau,\upsilon)\upsilon^k d\upsilon\right],$$

where $R(t,\tau)$ is the resolvent kernel for the integral equation (16).

In exactly the same way, let

$$X_2(z)=\begin{cases}\exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_{\partial D}\dfrac{\rho_2(\alpha_2^{-1}(\tau))d\tau}{\tau-z}\right\}, & z\in D^+,\\ z^{-\kappa_2}\exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_{\partial D}\dfrac{\rho_2(\tau)d\tau}{\tau-z}\right\}, & z\in D^-,\end{cases}\tag{22}$$

where ρ_2 is the unique solution of the Fredholm integral equation

$$(\mathbf{K}\rho_2)(t)=\log\{t^{-\kappa_2}G_2(t)\},\ \ t\in\partial D,\tag{23}$$

then it is the canonical function of the homogeneous Haseman BVP (8). Setting

$$\Psi_2(z)=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{\partial D}\dfrac{\chi_2(\alpha_2^{-1}(\tau))d\tau}{\tau-z}, & z\in D^+,\\ \dfrac{1}{2\pi i}\displaystyle\int_{\partial D}\dfrac{\chi_2(\tau)d\tau}{\tau-z}, & z\in D^-,\end{cases}\tag{24}$$

with the unique solution χ_2 of the Fredholm integral equation

$$(\mathbf{K}\chi_2)(t)=\left[X_2^+\left(\alpha_2(t)\right)\right]^{-1}g_2(t),\ \ t\in\partial D,\tag{25}$$

similarly, if and only if the conditions

$$\int_{\partial D}\tau^k\chi_2(\tau)d\tau=0,\ \ k=0,1,\cdots,-\kappa_2-1,\tag{26}$$

are satisfied, then the solution of problem (8) may be expressed as

$$V_2(z)=X_2(z)\left[\Psi_2(z)+q^{\alpha_2}_{\kappa_2-1}(z)\right],\tag{27}$$

where $q^{\alpha_2}_{\kappa_2-1}\in S^{\kappa_2-1}_{\alpha_2}$. If $\kappa_2>-1$, we also agree that the conditions (26) do not arise.

From (21) and (27), we obtain

$$V_1(z)=\frac{1}{z}\left\{X_1(z)\left[\Psi_1(z)+q^{\alpha_1}_{\kappa_1+1}(z)\right]-X_2(z)\left[\Psi_2(z)+q^{\alpha_2}_{\kappa_2-1}(z)\right]\right\}.\tag{28}$$

Setting

$$\begin{aligned}q^{\alpha_1}_{\kappa_1+1}(z)&=\sum_{j=0}^{\kappa_1+1}a_{1,j}\left[\omega_{\alpha_1}(z)-\omega_{\alpha_1}(0)\right]^j,\\ q^{\alpha_2}_{\kappa_2-1}(z)&=\sum_{j=0}^{\kappa_2-1}a_{2,j}\left[\omega_{\alpha_2}(z)-\omega_{\alpha_2}(0)\right]^j,\end{aligned}\tag{29}$$

in the following we call $a_{1,0}$ and $a_{2,0}$ the constant terms of $q_{\kappa_1+1}^{\alpha_1}$ and $q_{\kappa_2-1}^{\alpha_2}$ respectively. To make $V_1(z)$ analytic at $z=0$, we discuss now the problem in four cases.

Case 1: $\kappa_1+1\geq 0$ and $\kappa_2-1\geq 0$. When

$$a_{1,0}=\frac{1}{2\pi i}\int_{\partial D}\frac{c_0\chi_2(\alpha_2^{-1}(\tau))-\chi_1(\alpha_1^{-1}(\tau))}{\tau}d\tau+c_0a_{2,0} \tag{30}$$

with

$$c_0=\exp\left[\frac{1}{2\pi i}\int_{\partial D}\frac{\rho_2(\alpha_2^{-1}(\tau))-\rho_1(\alpha_1^{-1}(\tau))}{\tau}d\tau\right], \tag{31}$$

V_1 given by (28) is analytic at $z=0$.

Case 2: $\kappa_1+1\geq 0$ and $\kappa_2-1<0$. When

$$a_{1,0}=\frac{1}{2\pi i}\int_{\partial D}\frac{c_0\chi_2(\alpha_2^{-1}(\tau))-\chi_1(\alpha_1^{-1}(\tau))}{\tau}d\tau, \tag{32}$$

where c_0 is given in (31), V_1 given by (28) is analytic at $z=0$.

Case 3: $\kappa_1+1<0$ and $\kappa_2-1\geq 0$. When

$$a_{2,0}=-\frac{1}{2\pi i c_0}\int_{\partial D}\frac{c_0\chi_2(\alpha_2^{-1}(\tau))-\chi_1(\alpha_1^{-1}(\tau))}{\tau}d\tau, \tag{33}$$

where c_0 is given in (31), V_1 given by (28) is analytic at $z=0$.

Case 4: $\kappa_1+1<0$ and $\kappa_2-1<0$. When

$$\frac{1}{2\pi i}\int_{\partial D}\frac{c_0\chi_2(\alpha_2^{-1}(\tau))-\chi_1(\alpha_1^{-1}(\tau))}{\tau}d\tau=0 \tag{34}$$

with c_0 given in (31), which must be a condition of solvability, V_1 given by (28) is analytic at $z=0$.

To sum up the above discussion, we obtain the following result.

Theorem 1 *For the problem BSR_0 (4), there are four cases:*

1. *When $\kappa_1\geq -1$ and $\kappa_2\geq 1$, the problem BSR_0 (4) is solvable and its solution may be written as (6) with V_1 and V_2 given by (28) and (27) respectively, but the constant terms of $q_{\kappa_1+1}^{\alpha_1}$ and $q_{\kappa_2-1}^{\alpha_2}$ must satisfy the relation (30).*
2. *When $\kappa_1\geq -1$ and $\kappa_2<1$, if and only if the conditions of solvability (19) are satisfied, the problem BSR_0 (4) is solvable and its solution may be written as (6) with V_1 and V_2 given by (28) and (27) respectively, and the constant term of $q_{\kappa_1+1}^{\alpha_1}$ is given by (32).*

3. *When* $\kappa_1 < -1$ *and* $\kappa_2 \geq 1$, *if and only if the conditions of solvability* (26) *are satisfied, the problem* BSR_0 (4) *is solvable and its solution may be written as* (6) *with* V_1 *and* V_2 *given by* (28) *and* (27) *respectively, and the constant term of* $q_{\kappa_2-1}^{\alpha_2}$ *is given by* (33).

4. *When* $\kappa_1 < -1$ *and* $\kappa_2 < 1$, *if and only if the conditions of solvability* (19), (26) *and* (34) *are satisfied, the problem* BSR_0 (4) *is solvable and its solution may be written as* (6) *with* V_1 *and* V_2 *given by* (28) *and* (27) *respectively.*

3 Relative discussion

The expression of $V_1(z)$ may be obtained in another way. We illustrate the process with the case of both $\kappa_1 \geq -1$ and $\kappa_2 \geq 1$.

In fact, under this case (7) is equivalent to

$$\begin{cases} V_1^+(\alpha_1(t)) = G_1(t)V_1^-(t) + \left[G_1(t)\bar{t}V_2^-(t) - \overline{\alpha_1(t)}V_2^+(\alpha_1(t)) + g_1(t)\right], \\ \mathrm{Ord}(V_1, \infty) \leq 0, \end{cases} \tag{35}$$

where V_2 is determined by (27). The canonical function of the homogeneous BVP of (35) is

$$\widetilde{X}_1(z) = \begin{cases} \exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_{\partial D} \frac{\widetilde{\rho}_1(\alpha_1^{-1}(\tau))d\tau}{\tau - z}\right\}, & z \in D^+, \\ z^{-\kappa_1}\exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_{\partial D} \frac{\widetilde{\rho}_1(\tau)d\tau}{\tau - z}\right\}, & z \in D^-, \end{cases} \tag{36}$$

where $\widetilde{\rho}_1$ is the unique solution of the Fredholm integral equation

$$(\mathbf{K}\widetilde{\rho}_1)(t) = \log\{t^{-\kappa_1}G_1(t)\}, \ \ t \in \partial D. \tag{37}$$

Let

$$\widetilde{\Psi}_1(z) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\partial D} \frac{\widetilde{\chi}_1(\alpha_1^{-1}(\tau))d\tau}{\tau - z}, & z \in D^+, \\ \dfrac{1}{2\pi i}\displaystyle\int_{\partial D} \frac{\widetilde{\chi}_1(\tau)d\tau}{\tau - z}, & z \in D^-, \end{cases} \tag{38}$$

where $\widetilde{\chi}_1$ is the unique solution of the Fredholm integral equation

$$(\mathbf{K}\widetilde{\chi}_1)(t) = \left[\widetilde{X}_1^+(\alpha_1(t))\right]^{-1}\left[G_1(t)\bar{t}V_2^-(t) - \overline{\alpha_1(t)}V_2^+(\alpha_1(t)) + g_1(t)\right], \quad t \in \partial D. \tag{39}$$

Thus, the general solution of problem (35) may be expressed as

$$V_1(z) = \widetilde{X}_1(z)\left[\widetilde{\Psi}_1(z) + q_{\kappa_1}^{\alpha_1}(z)\right], \tag{40}$$

where $q_{\kappa_1}^{\alpha_1} \in S_{\alpha_1}^{\kappa_1}$. Therefore, we also have the following result.

Theorem 2 *When $\kappa_1 \geq -1$ and $\kappa_2 \geq 1$, the problem BSR_0 (4) is solvable and its solution may be written as* (6) *with V_1 and V_2 given by* (40) *and* (27) *respectively.*

Now we prove the equivalence of the solutions obtained in two ways.

Lemma 1 *The canonical functions X_1 given in* (12) *and $\widetilde{X}_1$ given in* (36) *satisfy the following relation*

$$\widetilde{X}_1(z) = \frac{\omega_{\alpha_1}(z) - \omega_{\alpha_1}(0)}{z} X_1(z). \tag{41}$$

Proof By (13) and (37),

$$\left(\mathbf{K}(\widetilde{\rho}_1 - \rho_1)\right)(t) = \log\left\{t\alpha_1^{-1}(t)\right\}, \ \ t \in \partial D,$$

which implies that

$$X(z) = \frac{\widetilde{X}_1(z)}{X_1(z)} = \begin{cases} \exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_{\partial D} \frac{(\widetilde{\rho}_1 - \rho_1)(\alpha_1^{-1}(\tau))d\tau}{\tau - z}\right\}, & z \in D^+ \\ \exp\left\{\dfrac{1}{2\pi i}\displaystyle\int_{\partial D} \frac{(\widetilde{\rho}_1 - \rho_1)(\tau)d\tau}{\tau - z}\right\}, & z \in D^- \end{cases}$$

satisfies

$$\alpha_1(t)X^+(\alpha_1(t)) = tX^-(t).$$

Also considering $\lim_{z\to\infty} X(z) = 1$, by the result in Section 1, we get $zX(z) = \omega_{\alpha_1}(z) - \omega_{\alpha_1}(0)$. This completes the proof.

Lemma 2 *When $\kappa_1 \geq -1$ and $\kappa_2 \geq 1$, V_1 given by* (28) *and* (40) *are equivalent if the constant terms in the polynomials $q_{\kappa_1+1}^1$ and $q_{\kappa_2-1}^2$ satisfy* (30).

Proof By Lemma 1 eqution (39) becomes

$$(\mathbf{K}\widetilde{\chi}_1)(t) = \frac{V_2^-(t)}{t\widetilde{X}_1^-(t)} - \frac{V_2^+(\alpha_1(t))}{\alpha_1(t)\widetilde{X}_1^+(\alpha_1(t))} + \frac{(\mathbf{K}\chi_1)(t)}{\omega_{\alpha_1}^+(\alpha_1(t)) - \omega_{\alpha_1}(0)},$$

which is equivalent to

$$\widetilde{\Psi}_1^+(\alpha_1(t))-\widetilde{\Psi}_1^-(t)=\frac{V_2^-(t)}{t\widetilde{X}_1^-(t)}-\frac{V_2^+(\alpha_1(t))}{\alpha_1(t)\widetilde{X}_1^+(\alpha_1(t))}+\frac{\Psi_1^+(\alpha_1(t))-\Psi_1^-(t)}{\omega_{\alpha_1}^+(\alpha_1(t))-\omega_{\alpha_1}(0)}. \tag{42}$$

Thus, we have

$$\begin{aligned}&\widetilde{\Psi}_1^+(\alpha_1(t))+\frac{V_2^+(\alpha_1(t))}{\alpha_1(t)\widetilde{X}_1^+(\alpha_1(t))}-\frac{\Psi_1^+(\alpha_1(t))}{\omega_{\alpha_1}^+(\alpha_1(t))-\omega_{\alpha_1}(0)}\\ =\;&\widetilde{\Psi}_1^-(t)+\frac{V_2^-(t)}{t\widetilde{X}_1^-(t)}-\frac{\Psi_1^-(t)}{\omega_{\alpha_1}^-(t)-\omega_{\alpha_1}(0)}.\end{aligned} \tag{43}$$

This leads to

$$\begin{aligned}&[\omega_{\alpha_1}(z)-\omega_{\alpha_1}(0)]\left[\widetilde{\Psi}_1(z)+\frac{V_2(z)}{z\widetilde{X}_1(z)}-\frac{\Psi_1(z)}{\omega_{\alpha_1}(z)-\omega_{\alpha_1}(0)}\right]\\ =&\begin{cases}\widetilde{q}_{\kappa_1}^{\alpha_1}(z), & \kappa_1\geq 0,\\ C, & \kappa_1=-1,\end{cases}\end{aligned} \tag{44}$$

where $\widetilde{q}_{\kappa_1}^{\alpha_1}\in S_{\alpha_1}^{\kappa_1}$. It is easy to know that, if $\kappa_1\geq 0$, the constant term in $\widetilde{q}_{\kappa_1}^{\alpha_1}$ satisfies (30) and if $\kappa_1=-1$ then the constant C is given by (30). Therefore, by (44) we get

$$\widetilde{\Psi}_1(z)=\left[\frac{\Psi_1(z)}{\omega_{\alpha_1}(z)-\omega_{\alpha_1}(0)}-\frac{V_2(z)}{z\widetilde{X}_1(z)}\right]+\begin{cases}\dfrac{\widetilde{q}_{\kappa_1}^{\alpha_1}(z)}{\omega_{\alpha_1}(z)-\omega_{\alpha_1}(0)}, & \kappa_1\geq 0,\\ \dfrac{C}{\omega_{\alpha_1}(z)-\omega_{\alpha_1}(0)}, & \kappa_1=-1.\end{cases}$$

Thus, for $\kappa_1\geq 0$, by (40),

$$\begin{aligned}V_1(z)&=\widetilde{X}_1(z)\left[\widetilde{\Psi}_1(z)+q_{\kappa_1}^{\alpha_1}(z)\right]\\ &=\frac{\omega_{\alpha_1}(z)-\omega_{\alpha_1}(0)}{z}X_1(z)\left\{\frac{\Psi_1(z)+\widetilde{q}_{\kappa_1}^{\alpha_1}(z)}{\omega_{\alpha_1}(z)-\omega_{\alpha_1}(0)}-\frac{V_2(z)}{z\widetilde{X}_1(z)}+q_{\kappa_1}^{\alpha_1}(z)\right\}\\ &=\frac{1}{z}\left\{X_1(z)\left[\Psi_1(z)+\left((\omega_{\alpha_1}(z)-\omega_{\alpha_1}(0))q_{\kappa_1}^{\alpha_1}(z)+\widetilde{q}_{\kappa_1}^{\alpha_1}(z)\right)\right]-V_2(z)\right\}\\ &=\frac{1}{z}\left\{X_1(z)\left[\Psi_1(z)+q_{\kappa_1+1}^{\alpha_1}(z)\right]-V_2(z)\right\},\end{aligned}$$

which is just the expression (28). When $\kappa_1=-1$, a similar discussion arrives at the same conclusion. This completes the proof.

Corollary 1 *In the case of* $\kappa_1 \geq -1$ *and* $\kappa_2 > 1$, *Theorem* 1 *and Theorem* 2 *are equivalent.*

Remark 2 Is must be pointed out that, for the other cases we may also obtain the same conclusion as Corollary 1 by similar discussions.

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Complex Hele-Shaw model. Local solvability for a doubly connected domain

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Abstract

The paper is devoted to the study of the Hele-Shaw moving boundary value problem in doubly-connected domains. It is reformulated as a couple of problems, namely, the Schwarz boundary value problem for a doubly-connected domain and an abstract Cauchy-Kovalevsky problem. Local existence and uniqueness of the classical solution to the Hele-Shaw problem is shown by using a variant of the Schwarz alternating method for the Schwarz problem and the Nirenberg-Nishida theorem for the Cauchy-Kovalevsky problem.

AMS Subject Classification: 76D27, 30E25, 35A10

Keywords: Hele-Shaw flows, doubly connected domain, classical solutions, Schwarz alternating method, abstract Cauchy-Kovalevsky problem

1 Introduction

The Hele-Shaw model as discussed in the classical paper by the british physicist H.S. Hele-Shaw [15] consists in the description of the dynamics of the frontier between two immiscible fluids in a narrow channel. In [38], [31] a mathematical model was derived for describing Hele-Shaw flows with a free boundary produced by injection/suction of fluid into/from a narrow channel supposing constant atmospheric pressure on the moving boundary. This model can be represented in the following form, where $f = f(z,t)$ is the Riemann mapping function from the unit disk G_1 onto the region occupied by fluid at the time t, and $Q(t)$ is the rate of injection/suction:

$$\mathrm{Re}\left(\frac{1}{z}\frac{\partial f}{\partial t}\overline{\frac{\partial f}{\partial z}}\right) = Q(t), \quad z \in \partial G_1, \ t \in [0,T), \tag{1}$$

$$f(z,0) = f_0(z), \quad z \in G_1, \tag{2}$$

$$f(0,t) = 0, \quad t \in [0,T). \tag{3}$$

This model commonly called complex Hele-Shaw model has been generalized in a number of direction (see e.g. [16], [36]). Local existence and uniqueness for (1)-(3) is shown in [38] under the additional condition $f'_z(0,t) > 0$. In the case of a rational initial mapping $f_0(z)$ an elementary proof of the local existence is proposed in [12]. In [30] the complex Hele-Shaw model is reformulated as an abstract Cauchy-Kovalevsky problem. The solution of the latter was found on the base of the Nirenberg-Nishida theorem [20], [21] (c.f. [23]). This approach is also used in [24], [28], [29] for analytical and numerical treatment of certain variants of the complex Hele-Shaw model. Among the result dealing with analytic solutions to the Hele-Shaw problem we have to mention the classical paper [25] in which a first collection of the exact solutions was constructed (c.f. also [17]).

The Hele-Shaw moving boundary value problem and its generalizations are studied on the base of certain weak or variational formulations starting from the results in [7], [13], exploiting the celebrated Baiocchi transformation [3]. These formulations of the Hele-Shaw model may be viewed also as an enthalpy formulation of the Stefan problem [8].

Several attempts were made to attack the Hele-Shaw problem in the case of when the fluid occupies a multiply connected region. In [14] an ill-posed moving boundary problem is studied for doubly connected domains in $\mathbb{R}^N$. A weak formulation of this problem is proposed, the

local in time solvability is shown. This problem is connected with Hele-Shaw model when $N = 2$ and with electro-chemical machining model as $N \leq 3$. Another model which also leads naturally to the moving boundary value problem in a multiply connected domain is that of freezing/solidification model (see e.g. [2]). In [32], [33] the Hele-Shaw problem for different type of multiply connected domains is reduced by applying the Cauchy transform to the functional equation in complex domains (for developing of this technique and other applications of the method of the functional equations see [19]).

The idea of the constructing a conformal mapping from a canonical n-connected region to the fluid domain, especially automorphic functions invariant with respect to the corresponding Schottky groups, is employed in [34] for the study of singularity-driven Hele-Shaw flows of a multiply connected fluid region.

In [5] the theory of algebraic curves and quadrature domains is used to construct exact solutions to the problem of the squeeze flow of multiply connected fluid domains in a Hele-Shaw cell.

In our paper we consider a generalization of the problem (1)-(3) to the case of doubly connected domains: *given a function $f_0(z)$, holomorphic and univalent in a neighbourhood of the domain*

$$G(0) \equiv \{z \in \mathbb{C} : |z - a_j| > r_j(0);\ |a_1 - a_2| > r_1(0) + r_2(0)\},$$

functions $Q_1(t), Q_2(t)$, continuous in a right-hand sided neighbourhood of $t = 0$, and a number $a \in [0, 2\pi)$, find $f(z,t)$, holomorphic and univalent in z in a neighbourhood of the domain

$$G(t) \equiv \{z \in \mathbb{C} : |z - a_j| > r_j(t);\ |a_1 - a_2| > r_1(t) + r_2(t)\},$$

$$\Gamma_j(t) \equiv \{z \in \mathbb{C} : |z - a_j| = r_j(t)\}, j = 1, 2,$$

and continuously differentiable in a right-hand sided neighbourhood of $t = 0$, satisfying the conditions

$$\mathrm{Re}\left(\frac{1}{z}\frac{\partial f}{\partial t}\overline{\frac{\partial f}{\partial z}}\right) = Q_j(t), \quad z \in \Gamma_j(t), j = 1, 2,\ t \in [0, T), \tag{4}$$

$$f(z, 0) = f_0(z), \quad z \in G(0), \tag{5}$$

$$\mathrm{Im}\left(\frac{1}{z}\frac{\partial f}{\partial t}\overline{\frac{\partial f}{\partial z}}\right)(z_0, t) = 0, \quad \text{for certain } z_0 \in G(t),\ t \in [0, T), \tag{6}$$

$$\int\limits_{\Gamma_1(t)} \left|\frac{\partial f}{\partial z}(\tau,t)\right|^{-2} \frac{Q_1(t)d\tau}{\tau-a_1} + \int\limits_{\Gamma_2(t)} \left|\frac{\partial f}{\partial z}(\tau,t)\right|^{-2} \frac{Q_2(t)d\tau}{\tau-a_2} = 0,\ t\in[0,T). \tag{7}$$

We follow here the strategy of the paper [29] in which an ill-posed Hele-Shaw problem for a simply connected domain was treated by reduction to an abstract Cauchy-Kovalevsky problem and a nonlinear Riemann-Hilbert-Poincaré problem (see also [30], [35]). The case of doubly connected domains leads to certain additional difficulties we have to overcome.

First, it is known (see [11]) that the conformal mapping of a doubly connected domain onto a canonical domain (in particular onto the exterior of two discs) depends on so called *conformal modulus* (in our case - the fraction of the radii of the circles $\Gamma_1(t), \Gamma_2(t)$). In our study we use an analytic dependence of the conformal modulus on small perturbation of $\Gamma_1(t), \Gamma_2(t)$ (see [18]).

Second, to get the unique conformal mapping of a doubly connected domain onto the canonical domain one has to add a (real) uniqueness condition which needs to have a physical sense. It can be proposed either in the form (6) or in the form

$$\arg\left(F_z'\right)(\infty,t) = 0, \ \text{ for a certain } z_0\in G(t),\ t\in[0,T), \tag{8}$$

where $F(z,t) \equiv \frac{1}{z}\frac{\partial f}{\partial t}\overline{\frac{\partial f}{\partial z}}$. Besides, the solution of the Schwarz boundary value problem for a multiply connected domain is in general a multi-valued function. Under condition (7) the solution becomes single-valued for each fixed $t\in[0,T)$.

Third, the problem (4)-(7) can be reduced to a couple of problems, namely, the Schwarz problem for the doubly connected domain $G(t)$ and an evolution equation to be solved successively. The first of these problems has a "global" and stationary character. The latter means that it should be solved for each fixed time instance, but the former means that we have to construct the solution in the whole domain. Moreover, we need to show the possibility to continue the obtained solution analytically into a neighbourhood of the initial domain and regularity dependence of the solution on the small perturbation of the boundary curves. These properties and the solution itself are obtained on the base of a variant of the Schwarz alternating method as proposed in [19] and by using the analyticity of the solution to the Schwarz problem with respect to the boundary curves shown in [26]. We have

to remark that in the contrary to the simply connected case there is no suitable exact formula for the solution to the Schwarz problem. The second problem, namely, the evolution equation, in the contrary, has a "local" character, i.e. is studied in a neighbourhood of each boundary curve. This problem is reduced to an abstract problem which is solved by using the following theorem which is a tool for study abstract nonlinear operator equations in a scale of Banach spaces.

Theorem 1 ([21]) *Let $\{\mathcal{B}_s, \|\cdot\|_s\}_{0<s\leq 1}$ be a scale of Banach spaces, i.e. a family of continuously embedded Banach spaces such that for all $0 < s' \leq s \leq 1$ the norm of the canonical embedding $\mathbf{I}_{s\to s'} : \mathcal{B}_s \to \mathcal{B}_{s'}$ is not greater than 1 ($\|\mathbf{I}_{s\to s'}\| \leq 1$).*

Let us consider the abstract Cauchy-Kovalevsky problem

$$\frac{dw}{dt} = \mathbf{L}(t, w), \quad w(0) = 0, \tag{9}$$

satisfying the following conditions in $\{\mathcal{B}_s, \|\cdot\|_s\}_{0<s\leq 1}$ (where C, K, R, and T be positive constants independent of s, s', t):

(i) *the nonlinear operator $\mathbf{L}(t, w)$ is continuous in t and maps*

$$[0, T] \times \{w \in \mathcal{B}_s : \|w\|_s \leq R\} \text{ into } \mathcal{B}_{s'} \text{ for all } 0 < s' \leq s \leq 1; \tag{10}$$

(ii) *the continuous function $\mathbf{L}(t, 0)$ satisfies*

$$\|\mathbf{L}(t, 0)\| \leq \frac{K}{1-s}, \text{ for all } 0 < s < 1; \tag{11}$$

(iii) *for all $0 < s' \leq s \leq 1$, $t \in [0, T)$, and w_1, w_2 belonging to the open ball $\{w \in \mathcal{B}_s : \|w\|_s < R\}$ we have*

$$\|\mathbf{L}(t, w_1) - \mathbf{L}(t, w_2)\|_{s'} \leq \frac{C}{s - s'}\|w_1 - w_2\|_s. \tag{12}$$

Under these assumptions there exists one and only one solution

$$w \in \mathcal{C}^1\left([0, a_0(1-s)), \mathcal{B}_s\right)_{0<s<1}, \ \|w\|_s < R,$$

where a_0 is a suitable positive constant.

Our paper is organized as follows. We start (Section 2) with reformulation of the initial problem (4)-(7) in an equivalent form, namely, in the form of the linear Schwarz boundary value problem for a doubly connected domain and an abstract Cauchy-Kovalevsky problem to

be solved successively. We also discuss the structure of the obtained couple of problems. In Section 3 a suitable scale of Banach spaces is constructed and its properties are studied. Solution of the Schwarz problem in a doubly connected domain is presented in Section 4. We conclude our investigation (Section 5) by the application of Theorem 1 to the Cauchy-Kovalevsky problem in the Banach scale defined in Section 3.

2 The structure of the problem

Let us consider the structure of problem (4)-(7). The equation (4) can be rewritten in the form ($j = 1, 2$)

$$\mathrm{Re}\left(\frac{1}{z}\frac{\partial f}{\partial t}(z,t)\left(\frac{\partial f}{\partial z}(z,t)\right)^{-1}\right) = \left|\frac{\partial f}{\partial z}(z,t)\right|^{-2} Q_j(t), \quad z \in \Gamma_j(t), \tag{13}$$

which have to be solved for each fixed $t \in [0, T)$. It is known [1] that under certain additional condition (e.g. under condition (6) or the condition

$$\mathrm{Im}\left(\frac{1}{z}\frac{\partial f}{\partial t}(z,t)\left(\frac{\partial f}{\partial z}(z,t)\right)^{-1}\right)\Big|_{z=z_0} = 0,$$

where z_0 is a point in $G(t)$) the problem (13) has a unique solution which is defined by means of the Schwarz operator $\mathbf{T}$ for the circular doubly connected domain. This solution is single valued if condition (7) is satisfied. Only few exact formulas are known for the Schwarz operator, namely the celebrated Villat's formula for a concentric annulus (see [1]) and a formula for multiply connected circular domain (a form of this formula is presented in [19]). The first form of the Schwarz operator is an integral operator with the Weierstrass zeta-function in the kernel, but in the last one the kernel is represented as a series in all elements of a certain Schottky group. Both forms are not suitable for all of our aims. Therefore our goal is to find an implicit formula for the Schwarz operator $\mathbf{T}$ for our domain $G(t)$, which will allow us to get the necessary properties of the solution.

Let us suppose that the operator $\mathbf{T}$ is already constructed. Then the equation (13) can be rewritten in the following equivalent form

$$\frac{\partial f}{\partial t}(z,t) = z\frac{\partial f}{\partial z}(z,t)\mathbf{T}\left(\left|\frac{\partial f}{\partial z}\right|^{-2} Q_j(t)\right), \quad z \in G(t), j = 1, 2. \tag{14}$$

For each fixed domain $G(t)$ we introduce the space $\mathcal{H}(G(t))\cap\mathcal{C}^{1,\alpha}(\overline{G(t)})$, as well as the spaces $\mathcal{H}(G(t))\cap\mathcal{C}^{\alpha}(\overline{G(t)})$, $\mathcal{H}(G(t))\cap\mathcal{C}(\overline{G(t)})$. Then the following lemma is an analog of Lemma 1 [30, p. 104], which can be proved by the change of the unknown function in the same manner as in [30].

Lemma 1 *Let* $f(z,t) \in \mathcal{C}^1\left([0,a_0),\mathcal{H}(G(t))\cap\mathcal{C}^{1,\alpha}(\overline{G(t)})\right)$ *for each* $t\in[0,a_0)$ *be a univalent function in* $G(t)$ *solving in* $\overline{G(t)}\times(0,a_0)$ *the problem* (14), (5) *(or equivalently* (4)-(7)*).*

Then the function

$$\phi(z,t)\equiv\left(\frac{\partial f}{\partial z}(z,t)\right)^{-1}\in\mathcal{C}^1\left([0,a_0),\mathcal{H}(G(t))\cap\mathcal{C}^{\alpha}(\overline{G(t)})\right)$$

is a solution to the problem

$$\begin{aligned}\frac{\partial\phi}{\partial t}(z,t) &= z\mathbf{T}_{G(t)}(\phi)(z,t)\phi(z,t)\frac{\partial}{\partial z}\phi(z,t)- \qquad (15)\\ &- \phi(z,t)\frac{\partial}{\partial z}\left(z\mathbf{T}_{G(t)}(\phi)(z,t)\right),\quad (z,t)\in G(t)\times(0,a_0),\end{aligned}$$

$$\phi(z,0)=\phi_0(z)\equiv\left(\frac{\partial f_0}{\partial z}(z)\right)^{-1},\quad z\in\overline{G(t)}, \qquad (16)$$

where $\phi(z,t)\neq 0$*, and* $\mathbf{T}_{G(t)}(\phi)$ *denotes the nonlinear operator*

$$\mathbf{T}_{G(t)}(\phi)\equiv\mathbf{T}\left(|\phi|^2\,Q(t)\right). \qquad (17)$$

Conversely, if $\phi(z,t)\in\mathcal{C}^1\left([0,a_0),\mathcal{H}(G(t))\cap\mathcal{C}^{\alpha}(\overline{G(t)})\right)$ *is a solution to* (15)-(16) *such that* $\phi(z,t)\neq 0$ *in* $\overline{G(t)}\times[0,a_0)$*. Then* $f(z,t)=\int\limits_0^z\frac{d\zeta}{\phi(\zeta,t)}$ *belongs to* $\mathcal{C}^1\left([0,a_0),\mathcal{H}(G(t))\cap\mathcal{C}^{1,\alpha}(\overline{G(t)})\right)$ *and represents a locally univalent solution to the problem (14), (5) (or equivalently (4)-(7)) in* $\overline{G(t)}\times(0,a_0)$*.*

By this lemma we reduce the starting problem (4)-(7) to the problem (15)-(16) which is a scale-type problem. Thus it remains to find a way how to interpret the problem (15)-(16) as a special case of (9). In Sec. 5 we will prove the existence of the solution $\phi(z,t)$ to the problem (15)-(16) which imply the following properties of the solution $f(z,t)$:

a) the family $f(z,t)$ belongs to the spaces $\mathcal{H}(G(t')) \cap \mathcal{C}^{1,\alpha}(\overline{G(t')})$ in the domains

$$G(t') \equiv \left\{ z \in \mathbb{C} : |z - a_j| > r_j(t');\, |a_1 - a_2| > r_1(t') + r_2(t') \right\},$$

which are situated in a neighbourhood of $G(t)$;

b) $f(z,t)$ is univalent for small t in the above said neighbourhood.

3 Scale of Banach spaces

Let us introduce analogously to [29] the following space of holomorphic functions (cf. [35]). In what follows we will assume for determination that $Q_1(t), Q_2(t)$ are negative in a right-hand sided neighbourhood of $t = 0$:

$$Q_1(t), Q_2(t) < 0, \quad t \in [0, T). \tag{18}$$

Such assumption corresponds to the shrinkage of both holes in the Hele-Shaw cell. All other situations can be considered analogously.

Let us fix the constants $r_{i,j}\, i, j = 1, 2, r_j(0) > r_{1,j} > r_{2,j} > 0$, a positive number $b > 0$ and introduce the parameter $s \in (0, 1)$. Denote by $\mathcal{H}\left(G_{(s)}\right)$ the space of functions, holomorphic in the following circular doubly connected domain

$$G_{(s)} \equiv \{ z \in \mathbb{C} : |z - a_j| > r_{1,j} - s(r_{1,j} - r_{2,j}), j = 1, 2 \}.$$

Then we define the space

$$\mathcal{B} := \Big\{ g = g(z,t) \in \bigcup_{0<s<1} \mathcal{C}\left([0, b(1-s)), \mathcal{H}(G_{(s)}) \cap \mathcal{C}^{1,\alpha}(\overline{G}_{(s)})\right) :$$

$$\|g\|_{\mathcal{B}} = \max \left\{ \sup_{s\in(0,1), h<b(1-s)} \max_{t\in[0,h]} \|g(\cdot,t)\|_{\mathcal{C}^{\alpha}(\overline{G}_{(s)})};\right.$$

$$\left. \sup_{s\in(0,1), t<b(1-s)} \left\| \frac{\partial g}{\partial z}(\cdot,t) \right\|_{\mathcal{C}^{\alpha}(\overline{G}_{(s)})} \left(1 - \frac{t}{b(1-s)}\right)^{\frac{1}{2}} \right\} < \infty \Big\}.$$

The following lemmas are simply generalization of [29, Lemmas 1,2].

Lemma 2 *The function space $\mathcal{B}$ is a Banach space.*

Proof Let $\{g_k\}$ be a Cauchy sequence from $\mathcal{B}$. Then it is a Cauchy sequence in the spaces $\mathcal{C}\left([0,h],\mathcal{H}(G_{(s)})\cap\mathcal{C}^{\lambda}(\overline{G}_{(s)})\right)$ for each $h<b(1-s)$. It follows from the completeness of every of the latter spaces that there exists a uniquely determined limit element $g=g_{(h,s)}\in\mathcal{C}\left([0,h],\mathcal{H}(G_{(s)})\cap\mathcal{C}^{\alpha}(\overline{G}_{(s)})\right)$. This element satisfies the estimate $\max_{t\in[0,h]}\|g_{(h,s)}\|_{\mathcal{C}^{\alpha}(\overline{G}_{(s)})}\leq\|g_{k_0}\|_{\mathcal{B}}+\varepsilon$ for a suitable index k_0 uniformly for all $h<b(1-s)$ and $s\in(0,1)$. The limit element of our Cauchy sequence $\{g_k\}$ is the function $g=g(z,t)$ with the property that g coincides with $g_{(h,s)}$ for all $h<b(1-s), s\in(0,1)$. Consequently

$$\sup_{s\in(0,1),h<b(1-s)}\max_{t\in[0,h]}\|g\|_{\mathcal{C}^{\alpha}(\overline{G}_{(s)})}<+\infty.$$

If $h<b(1-s)$ for a certain $s\in(0,1)$, then there exists a larger number $p\in(s,1)$ such that $h<b(1-p)$, too. Using Cauchy's integral formula for g in $\overline{G}_{(p)}$

$$g(z,t)=-\frac{1}{2\pi i}\int\limits_{\Gamma^1_{(p)}}\frac{g(\tau,t)}{\tau-z}d\tau-\frac{1}{2\pi i}\int\limits_{\Gamma^2_{(p)}}\frac{g(\tau,t)}{\tau-z}d\tau+C,$$

where $\Gamma^1_{(p)},\Gamma^2_{(p)}$ are boundary circles of the domain $G_{(p)}$, C is a constant, and estimating $\frac{\partial g}{\partial z}$ in $\overline{G}_{(s)}$ gives

$$g\in\mathcal{C}\left([0,b(1-s)),\mathcal{H}(G_{(s)})\cap\mathcal{C}^{1,\alpha}(\overline{G}_{(s)})\right).$$

Finally,

$$\begin{aligned}&\left\|\frac{\partial g}{\partial z}\right\|_{\mathcal{C}^{\alpha}(\overline{G}_{(s)})}\left(1-\frac{t}{b(1-s)}\right)^{\frac{1}{2}}\quad\leq\\ \leq&\left\|\frac{\partial g_{k_0}}{\partial z}\right\|_{\mathcal{C}^{\alpha}(\overline{G}_{(s)})}\left(1-\frac{t}{b(1-s)}\right)^{\frac{1}{2}}\quad+\quad\varepsilon\leq\|g_{k_0}\|_{\mathcal{B}}+\varepsilon\end{aligned}$$

gives the property

$$\sup_{s\in(0,1),t<b(1-s)}\left\|\frac{\partial g}{\partial z}(\cdot,t)\right\|_{\mathcal{C}^{\alpha}(\overline{G}_{(s)})}\left(1-\frac{t}{b(1-s)}\right)^{\frac{1}{2}}<\infty,$$

i.e. $g\in\mathcal{B}$.

Lemma 3 *The function space $\mathcal{B}$ is an algebra with*

$$\|g\cdot h\|_{\mathcal{B}}\leq 2\|g\|_{\mathcal{B}}\|h\|_{\mathcal{B}}\quad\textit{for all}\quad g,h\in\mathcal{B}.$$

Proof We know that $g, h \in \mathcal{C}\left([0, b(1-s)), \mathcal{H}(G_{(s)}) \cap \mathcal{C}^{1,\alpha}(\overline{G}_{(s)})\right)$.

The statement follows from the inequalities

$$\|g \cdot h\|_{\mathcal{C}^\alpha(\overline{G}_{(s)})} \leq \|g\|_{\mathcal{C}^\alpha(\overline{G}_{(s)})} \cdot \|h\|_{\mathcal{C}^\alpha(\overline{G}_{(s)})},$$

$$\left\|\frac{\partial}{\partial z}(g \cdot h)\right\|_{\mathcal{C}^\alpha(\overline{G}_{(s)})} \leq \left\|\frac{\partial}{\partial z} g\right\|_{\mathcal{C}^\alpha(\overline{G}_{(s)})} \|h\|_{\mathcal{C}^\alpha(\overline{G}_{(s)})} + \\ + \|g\|_{\mathcal{C}^\alpha(\overline{G}_{(s)})} \left\|\frac{\partial}{\partial z} h\right\|_{\mathcal{C}^\alpha(\overline{G}_{(s)})},$$

and the definition of $\|g\|_{\mathcal{B}}$ and $\|h\|_{\mathcal{B}}$.

In the case of simply connected domains the pre-image of the domains occupied by one of the fluids of the Hele-Shaw flow can be fixed (e.g. in the unit disc). This is not the case when the domain is doubly connected. For each fixed t one can choose two discs as pre-images of the "holes" in the Hele-Shaw cell. With time changing the ratio of the radii of these discs are changing too (one can suppose that one of these radii is constant but the other one is changing; alternatively, both radii can be supposed to change).

Later in the paper we need also the following result.

Lemma 4 *Let*

$$\mathcal{Z} \equiv \left\{\zeta \in C^1(\mathbb{T}, \mathbb{C}) : \inf\left\{\left|\frac{\zeta(s) - \zeta(t)}{s - t}\right| : s, t \in \mathbb{T}, s \neq t\right\} > 0\right\}$$

be the set of simple curves ζ of class $\mathcal{C}^1(\mathbb{T}, \mathbb{C})$.

Then the operator which assigns to each pair of simple smooth curves (ζ^1, ζ^2) from

$$\boldsymbol{\mathcal{Z}} \equiv \left\{\boldsymbol{\zeta} \equiv (\zeta^1, \zeta^2) \in \mathcal{Z}^2 : \zeta^1(\mathbb{T}) \subset \mathbb{E}[\zeta^2], \zeta^2(\mathbb{T}) \subset \mathbb{E}[\zeta^1]\right\}$$

the conformal modulus of the corresponding doubly connected domain is real analytic locally near any fixed pair $(\zeta_0^1, \zeta_0^2) \in \boldsymbol{\mathcal{Z}}$.

The proof in the case of an annulus is given in [18, Theorem 6.1] on the base of the Implicit Function Theorem. The proof in the case of any other doubly connected domain follows from conformal equivalence under Möbius transformation.

4 Schwarz alternating method

In order to get the corresponding properties of the solution to the Schwarz problem we introduce the following notations and results following mainly [19]. We formulate them in the case of an arbitrary circular n-connected domain.

Let $\mathbb{D} \equiv \widehat{\mathbb{C}} \setminus \left(\bigcup_{k=1}^{n} \operatorname{cl} \mathbb{D}_k \right)$ be a circular plane multiply connected domain, $\mathbb{D}_k \equiv \{z \in \mathbb{C} : |z - a_k| < r_k\}$. Let the formula

$$z^*_{(k_m,k_{m-1},\ldots,k_1)} = \left(z^*_{(k_{m-1},\ldots,k_1)} \right)^*_{(k_m)} \tag{19}$$

define successive symmetries with respect to the circles $\mathbb{T}_{k_1} = \partial \mathbb{D}_{k_1}, \ldots,$ $\mathbb{T}_{k_m} = \partial \mathbb{D}_{k_m}$. If in the sequence $k_m, k_{m-1}, \ldots, k_1$ no two neighbouring indices are equal then the number m is called the *level* of the mapping $z \mapsto z^*_{(k_m,k_{m-1},\ldots,k_1)}$. When m is even, these mappings are Möbius transformations. When m is odd, these mappings are anti-Möbius transformations, i.e. Möbius with respect to $\overline{z}$. The above described successive symmetries form for each fixed domain $\mathbb{D}$ the so-called Schottky group $\mathcal{K}$ of symmetries (see [9]). In the following we denote by $\mathcal{G}$ the subgroup of all but not identity elements of $\mathcal{K}$ of even level, and by $\mathcal{F}$ the family of all elements of $\mathcal{K}$ of odd level.

The following theorems represent the solution $\Psi(z) \equiv \mathbf{T}(\mathbb{D}, \mathbf{q})(z)$ to the Schwarz problem

$$\begin{cases} \operatorname{Re} \Psi(t) &= \mathbf{q}(t), \quad t \in \partial \mathbb{D}, \\ \operatorname{Im} \Psi(z_0) &= 0, \end{cases} \tag{20}$$

for a multiply connected circular domain $\mathbb{D}$ in the above introduced notations, where $\mathbf{q} \equiv (q_1, \ldots, q_n)$, $q_k : \mathbb{T}_k \to \mathbb{C}$ are given real-valued functions, $z_0 \in \mathbb{D}$ is a given point inside $\mathbb{D}$. We denote by $\mathbf{T}(\mathbb{D}, \mathbf{q})(\cdot) : \mathbf{q} \mapsto \Psi$ the Schwarz operator of the domain $\mathbb{D}$ with density $\mathbf{q}$.

Theorem 2 [19, Theorem 4.11] *The Schwarz operator* $\mathbf{T}(\mathbb{D}, \mathbf{q})(\cdot)$ *of a multiply connected circular domain* $\mathbb{D} \equiv \widehat{\mathbb{C}} \setminus \left(\bigcup_{k=1}^{n} \operatorname{cl} \mathbb{D}_k \right)$ *has the form* $\mathbf{T}(\mathbb{D}, \mathbf{q})(z) \equiv \Psi(z)$

$$\Psi(z) = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\mathbb{T}_k} q_k(\zeta) \left\{ \sum_j{}'' \left[\frac{1}{\zeta - \gamma_j(w)} - \frac{1}{\zeta - \gamma_j(z)} \right] + \right.$$

$$+\left(\frac{r_k}{\zeta-a_k}\right)^2 \sum_j{}' \left[\frac{1}{\overline{\overline{\zeta-\gamma_j\left(\overline{z}\right)}}}-\frac{1}{\overline{\overline{\zeta-\gamma_j\left(\overline{w}\right)}}}\right]-\frac{1}{\zeta-z}\Bigg\}d\zeta \qquad (21)$$

$$+\frac{1}{2\pi i}\sum_{k=1}^{n}\int_{\mathbb{T}_k} q_k(\zeta)\frac{\partial A}{\partial \nu}(\zeta)d\sigma+\sum_{m=1}^{n} A_m\left[\log\left(z-a_m\right)+\psi_m(z)\right]+i\varsigma,$$

where

$$A_m := \frac{1}{2\pi i}\sum_{k=1}^{n}\int_{\mathbb{T}_k} q_k(\zeta)\frac{\partial \alpha_m}{\partial \nu}(\zeta)d\sigma,\ m=1,2,...,n,$$

$A(\zeta)$ *has the form*

$$A(\zeta)=\sum_{m=1}^{n-1}\mathrm{Re}\left[\overline{\phi_m\left(\left(w^*_{(n)}\right)^*_{(m)}\right)}-\overline{\phi_m\left(w^*_{(m)}\right)}\right]+\log\left|\zeta-w^*_{(n)}\right|$$

$$-\alpha_n(\zeta)\log r_n-\sum_{m=1}^{n-1}\alpha_m(\zeta)\log\left|w^*_{(n)}-a_m\right|, \qquad (22)$$

the harmonic measure corresponding to the m-th circle $\alpha_m(\zeta)$ *has the form*

$$\alpha_l\left(z\right)=\sum_{m=1,m\neq l}^{n} A_m\left[\mathrm{Re}\,\psi_m\left(z\right)+\log\left|z-a_m\right|\right]+A, \qquad (23)$$

$\psi_m(z)$ *are given as*

$$\psi_m\left(z\right)=\log\left[\prod_{j=1,\ j\neq m}^{\infty}\psi_m^j\left(z\right)\right], \qquad (24)$$

where

$$\psi_m^j\left(z\right)=\begin{cases}\frac{\gamma_j(z)-a_m}{\gamma_j(w)-a_m},\ \text{if } \gamma_{\mathrm{j}}\in\mathcal{G},\\ \\ \overline{\frac{\gamma_j(\overline{w})-a_m}{\gamma_j(\overline{z})-a_m}},\ \text{if } \gamma_{\mathrm{j}}\in\mathcal{F}.\end{cases}$$

ς *is an arbitrary real constant,* $\sum'$ *is taken over all* j *with* γ_j *of odd level (* $\gamma_j\in\mathcal{F}$*), and* $\sum''$ *for all even level* γ_j $(\gamma_j\in\mathcal{G})$*. The series converges uniformly in each compact subset of* $\mathrm{cl}\,\mathbb{D}\backslash\{\infty\}$*.*

The single-valued part of the Schwarz operator appears in the modified Dirichlet problem:

$$\mathrm{Re}\,\Psi\left(s\right)=q_k(s)+c_k,\ t\in\mathbb{T}_k,\ k=1,2,...,n, \qquad (25)$$

where $q_k \in \mathcal{C}(\partial\mathbb{D}_k)$, $k = 1, \ldots, n$, is a given function, c_k are undetermined real constants. If one of the constants c_k is fixed arbitrarily, then the remaining ones are determined uniquely and $\Psi(z)$ is determined up to an arbitrary additive purely imaginary constant. Thus we have

Theorem 3 [19, Theorem 4.12] *The single-valued part of the Schwarz operator* $\mathbf{T}_{single}(\mathbb{D}, \mathbf{q})(\cdot)$ *of a multiply connected circular domain* $\mathbb{D} \equiv \widehat{\mathbb{C}} \setminus \left(\bigcup_{k=1}^{n} \operatorname{cl} \mathbb{D}_k \right)$ *corresponding to the modified Dirichlet problem (25) has the form*

$$\begin{aligned} &\mathbf{T}_{single}(\mathbb{D}, \mathbf{q})(z) \equiv \Psi(z) \\ &= \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\mathbb{T}_k} (q_k(\zeta) + c_k) \left\{ \sum_j{}'' \left[\frac{1}{\zeta - \gamma_j(w)} - \frac{1}{\zeta - \gamma_j(z)} \right] \right. \\ &\left. + \left(\frac{r_k}{\zeta - a_k} \right)^2 \sum_j{}' \left[\frac{1}{\overline{\zeta - \gamma_j(\overline{z})}} - \frac{1}{\overline{\zeta - \gamma_j(\overline{w})}} \right] - \frac{1}{\zeta - z} \right\} d\zeta \qquad (26) \\ &+ \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\mathbb{T}_k} q_k(\zeta) \frac{\partial A}{\partial \nu}(\zeta) d\sigma + i\varsigma. \end{aligned}$$

One of the real constants c_k *can be fixed arbitrarily; the remaining ones are determined uniquely from the linear algebraic system*

$$\sum_{k=1}^{n} \int_{\mathbb{T}_k} (q_k(\zeta) + c_k) \frac{\partial \alpha_m}{\partial \nu}(\zeta) d\sigma = 0, \ m = 1, 2, ..., n-1. \qquad (27)$$

By applying the properties of the Cauchy type operators (see e.g. [10]) and the successive symmetries as above introduced (see e.g. [19]) one can establish the following result.

Theorem 4 *The Schwarz operator* $\mathbf{T}(\mathbb{D}, \mathbf{q})(\cdot)$ *(as well as the single-valued Schwarz operator* $\mathbf{T}_{single}(\mathbb{D}, \mathbf{q})(\cdot)$*) of a multiply connected circular domain* $\mathbb{D} \equiv \widehat{\mathbb{C}} \setminus \left(\bigcup_{k=1}^{n} \operatorname{cl} \mathbb{D}_k \right)$ *is a bounded operator from* $\prod_{k=1}^{n} \mathcal{C}^{m,\alpha}(\mathbb{T}_k, \mathbb{R})$ *to* $\prod_{k=1}^{n} \mathcal{H}(\operatorname{ext} \mathbb{T}_k, \mathbb{C}) \bigcap \mathcal{C}^{m,\alpha}(\mathbb{T}_k, \mathbb{C})$.

Proof From an induction argument follows that it suffices to prove the statement in the case of a doubly connected domain (i.e. for $n = 2$) and for $m = 1$. The above Theorems 2, 3 give then the existence of the solution to the corresponding problem. The form of the Schwarz operators (see (21), (26)) does not allow to obtain boundedness of these operators as well as their regularity. Thus we use another approach.

Let us represent the solutions to the problem

$$\operatorname{Re}\Psi\left(s\right) = q_k(s),\ t \in \mathbb{T}_k,\ k = 1, 2, \tag{28}$$

in the following form

$$\Psi(z) = \Psi_1(z) + \Psi_2(z),\ \ z \in \operatorname{cl}\mathbb{D}, \tag{29}$$

where the functions Ψ_k, $k = 1, 2$, are holomorphic in the domains $\operatorname{ext}\mathbb{T}_k \equiv \{z \in \mathbb{C} : |z - a_k| > r_k\}$, continuous in $\operatorname{cl}\operatorname{ext}\mathbb{T}_k \equiv \{z \in \mathbb{C} : |z - a_k| \geq r_k\}$. Then the first equation of (28) can be rewritten as

$$\operatorname{Re}\Psi_1\left(s\right) + \operatorname{Re}\Psi_2\left(s^*_{(1)}\right) = q_1(s),\ \ |s - a_1| = r_1, \tag{30}$$

where $s^*_{(1)}$ is the point symmetric to s with respect to the first circle $\mathbb{T}_1$ (thus $s^*_{(1)} = s$). By applying to the boundary condition (30) the Schwarz operator of the exterior of the first circle, which is determined uniquely up to the real constant ς_1 by the following formula

$$\begin{aligned}\mathbf{T}_1 q_1(z) &\equiv \mathbf{T}(\operatorname{ext}\mathbb{T}_1, q_1)(z) \equiv \\ &\equiv -\frac{1}{2\pi i}\int\limits_{|\tau - a_1| = r_1} \frac{\tau + z - 2a_1}{\tau - z}\,\frac{q_1(\tau)d\tau}{\tau - a_1} + i\varsigma_1,\end{aligned} \tag{31}$$

we arrive at the functional equation in the domain $\operatorname{ext}\mathbb{T}_1$

$$\Psi_1(z) = -\Psi_2(z^*_{(1)}) + \mathbf{T}_1 q_1(z),\ \ |z - a_1| \geq r_1. \tag{32}$$

Analogously we obtain the functional equation in the domain $\operatorname{ext}\mathbb{T}_2$

$$\Psi_2(z) = -\Psi_1(z^*_{(2)}) + \mathbf{T}_2 q_2(z),\ \ |z - a_2| \geq r_2, \tag{33}$$

where the operator $\mathbf{T}_2$ is defined by the formula

$$\begin{aligned}\mathbf{T}_2 q_2(z) &\equiv \mathbf{T}(\operatorname{ext}\mathbb{T}_2, q_2)(z) \equiv \\ &\equiv -\frac{1}{2\pi i}\int\limits_{|\tau - a_2| = r_2} \frac{\tau + z - 2a_2}{\tau - z}\,\frac{q_2(\tau)d\tau}{\tau - a_2} + i\varsigma_2,\end{aligned} \tag{34}$$

The relations (32), (33) constitute a system of functional equations in the class of holomorphic functions. The system (32), (33) can be solved by successive approximations. Let us consider now the method of successive approximations for (32), (33) from another point of view. We take zero-order approximation for (32), (33) in the form

$$\Psi_1^0(z) \equiv \mathbf{T}_1 q_1(z), \quad |z - a_1| \geq r_1.$$

The function $\Psi_1^0(z)$ satisfies the first boundary condition of (28), but possibly does not satisfy the second one. This function is defined and holomorphic in the exterior of the first circle. Thus we can take it as a zero-order approximation to the solution in the domain $\mathbb{D} \equiv \{z \in \mathbb{C} : |z - a_k| > r_k,\ k = 1, 2\}$, namely

$$\Psi^0(z) \equiv \Psi_1^0(z)_{|\mathbb{D}}, \quad z \in \mathbb{D}.$$

Substituting $\Psi_1^0(z)$ into the functional equation (33) we obtain the first order

$$\Psi_2^1(z) \equiv -\Psi_1^0(z_{(2)}^*) + \mathbf{T}_2 q_2(z), \quad |z - a_2| \geq r_2,$$

and respectively

$$\Psi^1(z) \equiv \Psi^0(z) + \Psi_2^1(z)_{|\mathbb{D}}, \quad z \in \mathbb{D}.$$

The last function does satisfy the second boundary condition of (28), but possibly does not the first one. Further

$$\Psi_1^2(z) \equiv -\Psi_2^1(z_{(1)}^*), \quad |z - a_1| \geq r_1,$$

and consequently

$$\Psi^2(z) \equiv \Psi^0(z) + \Psi^1(z) + \Psi_1^2(z)_{|\mathbb{D}}, \quad z \in \mathbb{D}.$$

Finally, an N-th approximation of the solution can be given as

$$\Psi^N(z) \equiv \Psi^0(z) + \Psi^1(z) + \ldots + \Psi^{N-1}(z) + \Psi_1^N(z)_{|\mathbb{D}}, \quad z \in \mathbb{D},$$
$$N \text{ an even number}, \tag{35}$$
$$\Psi^N(z) \equiv \Psi^0(z) + \Psi^1(z) + \ldots + \Psi^{N-1}(z) + \Psi_2^N(z)_{|\mathbb{D}}, \quad z \in \mathbb{D},$$
$$N \text{an odd number}. \tag{36}$$

The proof of the convergence of the sequence $\Psi^N(z)$ in the space $\mathcal{H}(\mathbb{D}) \bigcap$
$\mathcal{C}(\mathrm{cl}\,\mathbb{D})$ for each fixed pair of functions $\mathbf{q} \equiv (q_1, q_2) \in \mathcal{C}^{0,\alpha}(\mathbb{T}_1) \times \mathcal{C}^{0,\alpha}(\mathbb{T}_1)$

is presented in [19, Section 4.9]. The similar proof can be given in the case of the space $\mathcal{H}(\mathbb{D}) \bigcap \mathcal{C}^{1,\alpha}(\mathrm{cl}\,\mathbb{D})$ for a fixed pair of data $\mathbf{q} \equiv (q_1, q_2) \in \mathcal{C}^{1,\alpha}(\mathbb{T}_1) \times \mathcal{C}^{1,\alpha}(\mathbb{T}_1)$.

In order to show this we have to mention that the solution of the Schwarz problem for a doubly connected circular domain $\mathbb{D}$ can be rewritten in the form of a sum of two series

$$\Psi(z) \equiv \widetilde{\Psi^1(z)} + \widetilde{\Psi^2(z)},$$

where

$$\widetilde{\Psi^1(z)} \equiv \mathbf{T}_1 q_1(z) + \mathbf{T}_1 q_1(\alpha(z)) + \mathbf{T}_1 q_1(\alpha^2(z)) + \ldots,$$

$$\widetilde{\Psi^2(z)} \equiv \mathbf{T}_2 q_2(z^*_{(1)}) + \mathbf{T}_2 q_2(\beta(z)) + \mathbf{T}_2 q_2(\beta^2(z)) + \ldots,$$

$$\alpha(z) \equiv z^*_{(12)},\ \beta(z) \equiv z^*_{(21)}, \tag{37}$$

$$\alpha^n(z) \equiv \alpha(\alpha^{n-1}(z)),\ \beta^n(z) \equiv \beta(\beta^{n-1}(z)).$$

Both operators $\boldsymbol{\alpha} : z \mapsto \alpha(z)$, $\boldsymbol{\beta} : z \mapsto \beta(z)$ are compact in the space $\mathcal{H}(\mathbb{D}) \bigcap \mathcal{C}^{1,\alpha}(\mathrm{cl}\,\mathbb{D})$ (see [19, Subsection 4.9.3]). Hence it remains to prove that the operators $\mathbf{T}_1, \mathbf{T}_2$ are bounded as operators acting from $\mathcal{C}^{1,\alpha}(\mathbb{T}_1)$, $\mathcal{C}^{1,\alpha}(\mathbb{T}_2)$ to $\mathcal{H}(\mathbb{D}) \bigcap \mathcal{C}^{1,\alpha}(\mathrm{cl}\,\mathbb{D})$ respectively. This result is basically a restatement of two technical lemmas from [18, Lemmas 5.1, 5.2]. This completes the proof.

5 Application of the abstract Cauchy-Kovalevsky problem

In this Section we apply the Nirenberg-Nishida theorem (Theorem 1) to the abstract Cauchy-Kovalevsky problem (15)-(16), where the operator $\mathbf{T}_{G(t)}$ is defined by the formula (17). We follow here the ideas and machinery of the papers [30], [29]. We can formulate the main result of our paper in the following manner.

Theorem 5 [Main Theorem] *There exists an in general small interval of time* $[0, b)$ *such that the problem (15)-(16) has a uniquely determined solution* $\phi = \phi(z, t)$ *which has no zeros on* $\overline{G}_{(p)} \times [0, b)$ *and belongs to the space* $\mathcal{C}^1\left([0, b), \mathcal{H}(G_{(p)}) \cap \mathcal{C}^{1,\alpha}(\overline{G}_{(p)})\right)$. *The constant* p *determining the doubly connected domain* $G_{(p)}$ *is taken as in the definition of* $\mathcal{B}$.

Corollary 1 *There exists an in general small interval of time* $[0, b)$ *such that the problem (4)-(7) has a uniquely determined solution* f. *The function* $f = f(z, t)$ *is univalent with respect to* z *on* $G_{(p)}$ *for each* $t \in [0, b)$ *and belongs to the above constructed space* $\mathcal{C}^1\left([0, b), \mathcal{H}(G_{(p)}) \cap \mathcal{C}^{2,\alpha}(\overline{G}_{(p)})\right)$. *The constant* p *determining the doubly connected domain* $G_{(p)}$ *is taken as in the definition of* $\mathcal{B}$.

This corollary is a direct consequence of the theorem. Univalence of the function f follows from the properties of the initial function f_0, the choice of $r_{i,j}\, i, j = 1, 2; r_j(0) > r_{1,j} > r_{2,j} > 0$ (see definition of $\mathcal{B}$) and the chosen norm of $\mathcal{B}$ which is stronger than the sup-norm.

To prove the main theorem we need a number of auxiliary results which we present below. We apply Theorem 1 to the problem (15)-(16). Hence the operator $\mathbf{L}(t, \phi)$ coincides with the operator on the right-hand side of (15). Thus the operator $\mathbf{L}(t, \phi)$ contains mainly the combination of two operators: the differentiation operator ∂_z and the Schwarz operator $\mathbf{T}$ for a doubly connected circular domain applied to $|\phi|^2$.

To study the unbounded operator ∂_z in the scale $\mathcal{B}$ we use the following result which is based on an idea formulated in [37]

Lemma 5 *Let* $\phi \in \mathcal{B}$. *Then the operator*

$$\mathbf{A} : \phi \mapsto \int_0^t \partial_z \phi(\cdot, \tau) d\tau, \quad t \in [0, b(1-s)), \quad s \in (0, 1),$$

is a continuous operator mapping $\mathcal{B}$ *into itself, and satisfying the estimate*

$$\|\mathbf{A}\phi\|_{\mathcal{B}} \leq Cb\|\phi\|_{\mathcal{B}}, \tag{38}$$

where $C = C(r_{i,j})$, $i, j = 1, 2$.

This lemma is a restatement of [29, Lemma 3].

Let us now begin to establish the necessary properties of the operator $\mathbf{T}_{G(t)}(\phi) \equiv \mathbf{T}\left(|\phi|^2 Q_j(t)\right)$. In the previous section we have shown the boundedness of this operator in the corresponding spaces for each fixed domain $G(t)$. But we need in Theorem 1 the fulfillment of stronger conditions.

First of all, the function $\mathbf{T}\left(|\phi|^2 Q_j(t)\right)(z)$ has to be continued analytically in a neighbourhood of the initial doubly connected domain $G(0)$ (thus to preserve the analogous property of the "initial" function

ϕ_0). To see this we mention first that the Schwarz operator can be rewritten in the form

$$\mathbf{T}\left(|\phi|^2 Q_j(t)\right)(z) = \mathbf{T}\left(\phi(\tau,t)\overline{\phi\left(\frac{1}{\overline{\tau}},t\right)}Q_j(t)\right)(z). \tag{39}$$

If the function $\phi(z,t)$ can be continued in a domain bigger than $G(t)$ then the conjugate function $\tilde{\phi}(z,t) \equiv \overline{\phi\left(\frac{1}{\overline{\tau}},t\right)}$ possesses the "continuation" in a smaller domain. Therefore we introduce another function space different from $\mathcal{B}$.

We fix as before the constants $r_{i,j}\, i,j = 1,2; r_j(0) > r_{1,j} > r_{2,j} > 0$, a positive number $b > 0$ and introduce a parameter $s \in (0,1)$. Denote by $\mathcal{H}\left(A_{(s)}\right)$ the space of functions, holomorphic in the union of two annuli

$$A_{(s)} \equiv A^1_{(s)} \bigcup A^2_{(s)},$$

$$A^j_{(s)} \equiv \left\{z \in \mathbb{C} : r_{1,j} - s(r_{1,j} - r_{2,j}) < |z - a_j| < \frac{1}{r_{1,j} - s(r_{1,j} - r_{2,j})}\right\}.$$

Then we define the space

$$\mathcal{B}_a := \Big\{g = g(z,t) \in \bigcup_{0<s<1} \mathcal{C}\left([0, b(1-s)), \mathcal{H}(A_{(s)}) \cap \mathcal{C}^{1,\alpha}(\overline{A}_{(s)})\right) :$$

$$\|g\|_{\mathcal{B}} = \max\Bigg\{\sup_{s\in(0,1), h<b(1-s)} \max_{t\in[0,h]} \|g(\cdot,t)\|_{\mathcal{C}^\alpha(\overline{A}_{(s)})};$$

$$\sup_{s\in(0,1), t<b(1-s)} \left\|\frac{\partial g}{\partial z}(\cdot,t)\right\|_{\mathcal{C}^\alpha(\overline{A}_{(s)})} \left(1 - \frac{t}{b(1-s)}\right)^{\frac{1}{2}}\Bigg\} < \infty\Big\}.$$

It is not hard to see that the space $\mathcal{B}_a$ is a Banach space and a Banach algebra with respect to the usual multiplication of complex-valued functions. The following lemma is of a straightforward verification.

Lemma 6 *If ϕ belongs to $\mathcal{B}$, then the function $\tilde{\phi}(z,t) \equiv \overline{\phi\left(\frac{1}{\overline{\tau}},t\right)}$ as well as the product $\phi\tilde{\phi}$ belongs to $\mathcal{B}_a$. Besides,*

$$\|\tilde{\phi}\|_{\mathcal{B}_a} \le C\|\phi\|_{\mathcal{B}}.$$

Now all is prepared for the study of $\mathbf{T}_{G(t)}(\phi)$ on $\mathcal{B}$. Let

$$G_{\eta_1,\eta_2} \equiv \{z \in \mathbb{C} : |z - a_1| > \eta_1 r_1,\ |z - a_2| > \eta_2 r_2\}$$

be a neighbourhood of a doubly connected domain $G(0)$. Let the function v be in $\mathcal{C}\left([0,T),\mathcal{H}(G_{\eta_1,\eta_2})\cap\mathcal{C}(\overline{G}_{\eta_1,\eta_2})\right)$ for certain $\eta_1,\eta_2<1$. Then the function $\mathbf{T}_{G(t)}(v)\equiv\mathbf{T}(G(t),\mathbf{q}|v|^2)$ as defined in (17) belongs to $\mathcal{H}(G(t))$ for each $t\in[0,T)$. Moreover, the following lemma is valid.

Lemma 7 *For any $v\in\mathcal{C}\left([0,T),\mathcal{H}(G_{\eta_1,\eta_2})\cap\mathcal{C}(\overline{G}_{\eta_1,\eta_2})\right)$ the image $\mathbf{T}_{G(t)}(v)$ of the nonlinear operator $\mathbf{T}_{G(t)}$ possesses an analytic continuation into the bigger domain $G_{\eta_1^0,\eta_2^0}$ for certain $\eta_1^0\le\eta_1<1$, $\eta_2^0\le\eta_2<1$.*

Proof First we have to show that both integrals $\mathbf{T}_1$, $\mathbf{T}_2$ (see (31), (34)) meet this property. It suffices to prove this only for $\mathbf{T}_1$. Let us rewrite first $\mathbf{T}_1(\operatorname{ext}\mathbb{T}_1,Q_1|v|^2)$ in the form analogous to (39)

$$\mathbf{T}_1(\operatorname{ext}\mathbb{T}_1,Q_1|v|^2)(z)=\mathbf{T}_1(\operatorname{ext}\mathbb{T}_1,Q_1v(\tau,t)\overline{v\left(\frac{1}{\overline{\tau}},t\right)})(z).$$

Since $v\in\mathcal{C}\left([0,T),\mathcal{H}(G_{\eta_1,\eta_2})\cap\mathcal{C}(\overline{G}_{\eta_1,\eta_2})\right)$ then by Lemma 6 the functions $\tilde{v}=\overline{v\left(\frac{1}{\overline{\tau}},t\right)})$ and $\tilde{v}v$ possess an analytic continuation into an annulus $A^1_{(s)}$ for certain $0<s<1$. By uniqueness of the analytic continuation we can conclude that the operator $\mathbb{T}_1$ can be rewritten in the form

$$\mathbf{T}_1(\operatorname{ext}\mathbb{T}_1,Q_1|v|^2)(z)= \tag{40}$$

$$-\frac{1}{2\pi i}\int\limits_{|\tau-a_1|=\eta_1r_1}\frac{\tau+z-2a_1}{\tau-z}\frac{Q_1(t)v(\tau,t)\overline{v\left(\frac{1}{\overline{\tau}},t\right)})d\tau}{\tau-a_1}+i\varsigma_1,$$

for each $t\in[0,T_1)$ with certain $T_1>0$. The kernel of this integral is holomorphic in the exterior of the circle $|\tau-a_1|=\eta_1r_1$. It gives the desired result for the operator $\mathbf{T}_1$. A similar consideration can be applied to the operator $\mathbf{T}_2$.

The functions $\alpha(z)$ and $\beta(z)$ as defined in (37) are in fact Möbius transformations and are holomorphic outside at most one point. Therefore each term in the series $\widetilde{\Psi_1(z)}$, $\widetilde{\Psi_2(z)}$ has the analytic continuation properties as above formulated. The final result follows from the uniformity of the convergence of these series.

Lemma 8 *The operator*

$$\mathbf{T}_{G(t)}(\phi)\equiv\mathbf{T}\left(|\phi|^2\,Q(t)\right)$$

satisfies in $\mathcal{B}$ the inequalities

$$\left\|\mathbf{T}_{G_{\eta_1^0,\eta_2^0}}(\phi)\right\|_{\mathcal{B}} \le C_1\|\phi\|_{\mathcal{B}}^2, \tag{41}$$

$$\left\|\mathbf{T}_{G_{\eta_1^0,\eta_2^0}}(\phi_1) - \mathbf{T}_{G_{\eta_1^0,\eta_2^0}}(\phi_2)\right\|_{\mathcal{B}} \le C_1\|\phi_1-\phi_2\|_{\mathcal{B}}^2. \tag{42}$$

Moreover, the family of operators $\{\mathbf{T}_{G(t)}(\phi)\}_{t\in[0,T)}$ depends continuously on $t \in [0,t)$.

Proof We have to start again with the operator $\mathbf{T}_1$. By rewriting it in the form (40) we can use then [29, Lemma 5]. The final result follows then from the properties of the series $\widetilde{\Psi_1(z)}$, $\widetilde{\Psi_2(z)}$.

At last we need to show that the right-hand side of (15) does satisfy the conditions of Theorem 1. To do this we rewrite equations (15) in a suitable form

$$\begin{aligned}\frac{\partial\omega}{\partial t}(z,t) &\equiv \mathbf{L}_0(t,\omega) = z\mathbf{T}_{G_{\eta_1^0,\eta_2^0}}(\omega+\phi_0)(z,t)\frac{\partial}{\partial z}(\omega(z,t)+\phi_0(z)) \\ &- (\omega(z,t)+\phi_0(z))\frac{\partial}{\partial z}\left(z\mathbf{T}_{G_{\eta_1^0,\eta_2^0}}((\omega+\phi_0)(z,t)\right),\end{aligned}$$

$$\omega(z,0) \equiv \phi(z,0) - \phi_0(z) = 0.$$

Lemma 9 *The operator $\mathbf{L}_0$ satisfies in the scale $\{\mathcal{B}_s, \|\cdot\|_s\}_{0<s\le 1}$ of Banach spaces*

$$\{\mathcal{B}_s\}_{0<s\le 1} := \Big\{g = g(z,t) \in \mathcal{C}\left([0,b(1-s)), \mathcal{H}(G_{(s)}) \cap \mathcal{C}^{1,\alpha}(\overline{G}_{(s)})\right) :$$

$$\|g\|_{\mathcal{B}} = \max\Bigg\{\sup_{s\in(0,1),h<b(1-s)}\max_{t\in[0,h]}\|g(\cdot,t)\|_{\mathcal{C}^\alpha(\overline{G}_{(s)})};$$

$$\sup_{s\in(0,1),t<b(1-s)}\left\|\frac{\partial g}{\partial z}(\cdot,t)\right\|_{\mathcal{C}^\alpha(\overline{G}_{(s)})}\left(1-\frac{t}{b(1-s)}\right)^{\frac{1}{2}}\Bigg\} < \infty,\Big\}$$

the conditions (10)-(12) of Theorem 1.

Proof The property (10) follows from the second statement of Lemma 8. Further, Lemma 5 gives that the operator $\frac{\partial}{\partial z}$ is a bounded operator from $\mathcal{B}_s$ to $\mathcal{B}_{s'}$ and

$$\left\|\frac{\partial}{\partial z}\right\|_{s\to s'} \le \frac{1}{\min\{r_{1,1}-r_{2,1}; r_{1,2}-r_{2,2}\}(s-s')}. \tag{43}$$

In order to get the properties (11)-(12) we rewrite the difference $\mathbf{L}_0(t,\omega_1) - \mathbf{L}_0(t,\omega_2)$ in the form

$$\begin{aligned}
\mathbf{L}_0(t,\omega_1)&-\mathbf{L}_0(t,\omega_2) \equiv -(\omega_1-\omega_2)\frac{\partial}{\partial z}\left(z\mathbf{T}_{G_{\eta_1^0,\eta_2^0}}(\omega_1+\phi_0)(z,t)\right)\\
-(\omega_2+\phi_0)&\frac{\partial}{\partial z}\left(z(\mathbf{T}_{G_{\eta_1^0,\eta_2^0}}(\omega_1+\phi_0)(z,t)-\mathbf{T}_{G_{\eta_1^0,\eta_2^0}}(\omega_2+\phi_0)(z,t))\right)\\
&+z(\mathbf{T}_{G_{\eta_1^0,\eta_2^0}}(\omega_1+\phi_0)(z,t)-\mathbf{T}_{G_{\eta_1^0,\eta_2^0}}(\omega_2+\phi_0)(z,t))\frac{\partial}{\partial z}(\omega_1+\phi_0)\\
&+z\mathbf{T}_{G_{\eta_1^0,\eta_2^0}}(\omega_2+\phi_0)(z,t)\frac{\partial}{\partial z}(\omega_1-\omega_2).
\end{aligned}$$

Then inequality (11) follows from Lemma 8 and inequality (43).

In the same manner we can prove that

$$\begin{aligned}
&\|\mathbf{L}_0(t,0)\|_s \quad =\\
&\left\|\phi_0\frac{\partial}{\partial z}\left(z\mathbf{T}_{G_{\eta_1^0,\eta_2^0}}(\phi_0)(z,t)\right)-z\mathbf{T}_{G_{\eta_1^0,\eta_2^0}}(\phi_0)(z,t)\frac{\partial}{\partial z}\phi_0(z)\right\|_s \le \frac{K}{1-s}.
\end{aligned}$$

By applying Lemma 9 we obtain the existence and uniqueness of the local (in time) analytic solution to the Hele-Shaw problem for a doubly connected domain (as formulated in the Main Theorem).

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Computing porosity of cancellous bone using ultrasonic waves

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Abstract

Based on a model and an algorithm proposed previously by Buchanan, Gilbert, Wirgin and Xu [4], we numerically computed the dependence of the porosity of cancellous bones on the measured ultrasonic wave. The computation shows the bone porosity can be recovered very accurately. The numerical data show that the ultrasonic wave of higher frequency would produce a more accurate bone porosity.
AMS Subject Classification 2000: 35A05, 76B40
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1 Introduction

Biot developed a general theory for the ultrasonic propagation in fluid-saturated porous media [1]-[3]. McKelvie and Palmer [6], Williams [7], and Hosokawa and Otani [5] discussed the application of Biot's model for a poroelastic medium to cancellous bones. Recently, Buchanan, Gilbert, Wirgin and Xu [4] proposed a model for the determination of the parameters of cancellous bone using ultrasonic measurements. Following their model, we consider a piece of cancellous bone submerged in fluid, see Figure 1. Ultrasonic wave is produced from the transmitter at one side and received by a hydrophone at the other side of a bone. We apply Biot's system of compressional wave equations for a dispersive dissipative fluid-saturated porous medium in the time domain to this model. We use the reflection and transmission of ultrasonic waves in cancellous bone and use the information to determine the bone porosity. The numerical computation, based on known parameters, shows some encouraging results.

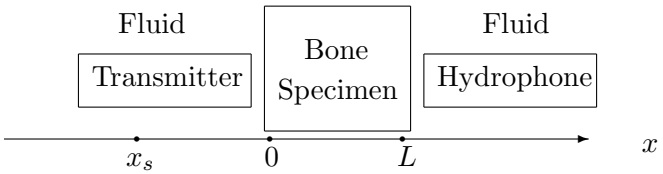


Figure 1. Experiment arrangement and one-dimensional computation.

2 The Biot model

We consider a one-dimensional dynamic model in this paper. In the bone ($0 < x < L$), cf. Figure 1, the ultrasonic wave satisfies the one-dimensional dynamic equations

$$\begin{aligned}
\frac{\partial}{\partial x}[\lambda e + Q\epsilon] &= \frac{\partial^2}{\partial t^2}(\rho_{11}u + \rho_{12}U) + b\frac{\partial}{\partial t}(u - U),\\
\frac{\partial}{\partial x}[Qe + R\epsilon] &= \frac{\partial^2}{\partial t^2}(\rho_{12}u + \rho_{22}U) - b\frac{\partial}{\partial t}(u - U),\\
e = \frac{\partial u}{\partial x}, &\qquad \epsilon = \frac{\partial U}{\partial x}.
\end{aligned} \tag{1}$$

Here, the motion of the frame and fluid within the bone are tracked by position vectors u and U respectively (Ref. [1]-[3]).

In the water, ($x < 0$ or $x > L$) let p_0 be the acoustic pressure and U_0 the displacement. If the transducer is located at $x = x_s$ with waveform $f(t)$, then the acoustic pressure obeys

$$\begin{aligned} \frac{\partial^2 p_0}{\partial t^2} - c_0^2 \frac{\partial^2 p_0}{\partial x^2} &= f(t)\delta(x - x_s), \\ \frac{\partial p_0}{\partial x} &= \rho_0 \frac{\partial^2 U_0}{\partial t^2}. \end{aligned} \tag{2}$$

On the interface ($x_0 = 0$ or $x_0 = L$), the displacement is continuous, and the normal stress and the pore stress in the bone is equal to the acoustic pressure in the fluid. Therefore, the displacements, pressure and stresses satisfy

$$\begin{aligned} U_0(x_0^-) &= \beta U(x_0^+) + (1-\beta)u(x_0^+), \\ p_0(x_0^-) &= \sigma_{xx}(x_0^+) + \sigma(x_0^+), \\ p_0(x_0^-) &= \sigma(x_0^+)/\beta. \end{aligned} \tag{3}$$

Here $x_0^- = 0^-$ or L^+, and $x_0^+ = 0^+$ or L^-, respectively. In (3), we recall that $\sigma_{xx} = \lambda e + Q\epsilon$, $e = \frac{\partial u}{\partial x}$, and $\epsilon = \frac{\partial U}{\partial x}$. There are six equations in (3) totally. The transient property of the wave implies that $\lim_{|x|\to\infty} U_0 = 0$.

There are seven parameters to be determined in the one-dimensional Biot model. Naturally, in a one-dimensional model, the complex frame shear modulus μ is not included. The other parameters λ, R and Q occurring in the constitutive equations are calculated from the measured or estimated values of the parameters given in Table 1(see references [4]-[7].)

The other parameters are computed according to the formulas

$$\begin{aligned} \lambda &= K_b - \frac{2}{3}\mu + \frac{(K_r - K_b)^2 - 2\beta K_r(K_r - K_b) + \beta^2 K_r^2}{D - K_b}, \\ R &= \frac{\beta^2 K_r^2}{D - K_b}, \\ Q &= \frac{\beta K_r\left((1-\beta)K_r - K_b\right)}{D - K_b}, \end{aligned} \tag{4}$$

where we set $\mu = 0$ and

$$D = K_r\left(1 + \beta\left(\frac{K_r}{K_f} - 1\right)\right).$$

Table 1. Parameters used in the Biot model

Symbol	Value	Parameter
ρ_f	950	Density of the pore fluid
ρ_r	1960	Density of frame material
K_b	$Re(K_b) = 1 \times 10^9$ $Im(K_b) = 0.1Re(K_b)/\pi = 3.18 \times 10^7$	Complex frame bulk modulus
K_f	2×10^9	Fluid bulk modulus
K_r	2×10^{10}	Frame material bulk modulus
β	0.75	Porosity
η	1.5	Viscosity of pore fluid
κ	1×10^{-8} or $\frac{\beta a^2}{4K}$ for $K = 5$	Permeability
α	1.0833 $1 - s(1 - 1/\beta)$ for $s = 1/4$	Structure constant
a	0.0015	Pore size parameter

In (2) ρ_{11} and ρ_{22} are density parameters for the solid and fluid, ρ_{12} is a density coupling parameter, and b is a dissipation parameter which depends on the wave frequency, ω.

$$\begin{aligned} \rho_{11} &= (1-\beta)\rho_r - \beta(\rho_f - m\beta), \\ \rho_{12} &= \beta(\rho_f - m\beta), \\ \rho_{22} &= m\beta^2, \\ b &= \frac{F\left(a\sqrt{\omega\rho_f/\eta}\right)\beta^2\eta}{\kappa}, \end{aligned} \tag{5}$$

where

$$m = \frac{\alpha\rho_f}{\beta},$$

and the multiplicative factor $F(\zeta)$, which was introduced by Biot to correct for the invalidity of the assumption of Poiseuille flow at high frequencies, is given by

$$F(\zeta) = \frac{1}{4}\frac{\zeta T(\zeta)}{1 - 2T(\zeta)/i\zeta},$$

where T is defined in terms of Kelvin functions

$$T(\zeta) = \frac{\mathrm{ber}'(\zeta) + i\mathrm{bei}'(\zeta)}{\mathrm{ber}(\zeta) + i\mathrm{bei}(\zeta)}.$$

3 Transient reflection and transmission of waves

Let $\hat{p}_0$, $\hat{U}_0$, $\hat{e}$, and $\hat{\epsilon}$ be the Fourier transforms of p_0, U_0, e, and ϵ, respectively, for the ω. Then from (2), the transformed equations in the water ($x < 0$ or $x > L$) become

$$\begin{aligned} -\omega^2 \hat{p}_0 - c_0^2 \frac{\partial^2 \hat{p}_0}{\partial x^2} &= \hat{f}\delta(x - x_s), \\ \frac{\partial \hat{p}_0}{\partial x} &= -\rho_0 \omega^2 \hat{U}_0. \end{aligned} \tag{6}$$

Correspondingly, in the bone ($0 < x < L$), the transformed equations from (1) become

$$\begin{aligned} \frac{\partial^2}{\partial x^2}[\lambda \hat{u} + Q\hat{U}] &= -\omega^2(\rho_{11}\hat{u} + \rho_{12}\hat{U}) + i\omega b(\hat{u} - \hat{U}), \\ \frac{\partial^2}{\partial x^2}[Q\hat{u} + R\hat{U}] &= -\omega^2(\rho_{12}\hat{u} + \rho_{22}\hat{U}) - i\omega b(\hat{u} - \hat{U}). \end{aligned} \tag{7}$$

On the interface, ($x_0 = 0$ or $x_0 = L$), from (3), we have the transformed transmission conditions now as

$$\begin{aligned} \hat{U}_0(x_0^-) = -\frac{\partial \hat{p}_0}{\partial x}(x_0^-) &= \beta \hat{U}(x_0^+) + (1-\beta)\hat{u}(x_0^+), \\ \hat{p}_0(x_0^-) &= (\lambda + Q)\frac{\partial \hat{u}}{\partial x}(x_0^+) + (Q+R)\frac{\partial \hat{U}}{\partial x}(x_0^+), \\ \hat{p}_0(x_0^-) &= \frac{1}{\beta}\left[Q\frac{\partial \hat{u}}{\partial x}(x_0^+) + R\frac{\partial \hat{U}}{\partial x}(x_0^+)\right]. \end{aligned} \tag{8}$$

Here, again, $x_0^- = 0^-$ or L^+, and $x_0^+ = 0^+$ or L^-, respectively.

Introducing the new unknowns $\hat{u}_x = \frac{\partial \hat{u}}{\partial x}$ and $\hat{U}_x = \frac{\partial \hat{U}}{\partial x}$, the system (7) can be decoupled into the matrix form

$$\frac{\partial}{\partial x}\begin{pmatrix} \hat{u} \\ \hat{U} \\ \hat{u}_x \\ \hat{U}_x \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ l_{11} & l_{12} & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{u} \\ \hat{U} \\ \hat{u}_x \\ \hat{U}_x \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned} l_{11} &= -\omega^2 a_{11} + i\omega b l_1, & l_{12} &= -\omega^2 a_{12} - i\omega b l_1, \\ l_{21} &= -\omega^2 a_{21} - i\omega b l_2, & l_{21} &= -\omega^2 a_{22} + i\omega b l_2, \end{aligned}$$

and

$$l_1 = \frac{Q+R}{\delta}, \qquad l_2 = \frac{Q+\lambda}{\delta},$$

$$a_{11} = \frac{R\rho_{11} - Q\rho_{12}}{\delta}, \qquad a_{12} = \frac{R\rho_{12} - Q\rho_{22}}{\delta},$$
$$a_{21} = \frac{\lambda\rho_{12} - Q\rho_{11}}{\delta}, \qquad a_{22} = \frac{\lambda\rho_{22} - Q\rho_{12}}{\delta},$$
$$\delta = \lambda R - Q^2.$$

The homogeneous system (9) can be easily solved to have the solution

$$\begin{pmatrix} \hat{u} \\ \hat{U} \\ \hat{u}_x \\ \hat{U}_x \end{pmatrix} = \mathbf{E}(x)\mathbf{C}, \tag{10}$$

where $\mathbf{E}(x) = \mathbf{E_0} e^{\mathbf{\Lambda_0} x}$, and $\mathbf{E_0}$ and $\mathbf{\Lambda_0}$ are the eigenvector and eigenvalue matrices of the 4×4 matrix in (9), and the constant vector $\mathbf{C} = (C_1, C_2, C_3, C_4)^T$ will be determined by the boundary conditions and the interface coupling.

In the water, for $x < 0$ or $x > L$, the transformed acoustic pressure satisfies

$$\frac{\partial^2 \hat{p}_0}{\partial x^2} + \frac{\omega^2}{c_0^2}\hat{p}_0 = -\frac{\hat{f}}{c_0^2} \cdot \delta(x - x_s),$$
$$\hat{p}_0 \sim e^{ik|x|}, \qquad |x| \to \infty,$$

where $k = \frac{\omega}{c_0}$ and c_0 is the sound speed in water. Therefore, for some constants $c_i, i = 1, ..., 4$, the transformed acoustic pressure has the form

$$\hat{p}_0(x) = \begin{cases} c_1 e^{-ikx}, & x < x_s, \\ c_2 e^{ikx} + c_3 e^{-ikx}, & x_s < x < 0, \\ c_4 e^{ikx}, & x > L. \end{cases} \tag{11}$$

Using the interface conditions (8), we have the following system of

linear equations, for $C_i, c_i, i = 1, 2, 3, 4$,

$$\begin{pmatrix} 1 & -e^{2ikx_s} & -1 & 0 & \mathbf{0} \\ 1 & e^{2ikx_s} & -1 & 0 & \mathbf{0} \\ 0 & -\frac{ik}{\rho_0\omega^2} & \frac{ik}{\rho_0\omega^2} & 0 & -\beta\mathbf{E_2}(0) - (1-\beta)\mathbf{E_1}(0) \\ 0 & 1 & 1 & 0 & -(\lambda+Q)\mathbf{E_3}(0) - (Q+R)\mathbf{E_4}(0) \\ 0 & 1 & 1 & 0 & -\frac{Q}{\beta}\mathbf{E_3}(0) - R\mathbf{E_4}(0) \\ 0 & 0 & 0 & -\frac{ike^{ikL}}{\rho_0\omega^2} & -\beta\mathbf{E_2}(L) - (1-\beta)\mathbf{E_1}(L) \\ 0 & 0 & 0 & e^{ikL} & -(\lambda+Q)\mathbf{E_3}(L) - (Q+R)\mathbf{E_4}(L) \\ 0 & 0 & 0 & e^{ikL} & -\frac{Q}{\beta}\mathbf{E_3}(L) - R\mathbf{E_4}(L) \end{pmatrix} \tag{12}$$

$$\times \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\hat{f}e^{ikx_s}}{ikc_0^2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

4 Numerical tests

In the paper [4], it is shown that from the measured data by the hydrophone (see Figure 1), the porosity of cancellous bones is uniquely determined locally. In this section, we will compute the dependence of the porosity β on the measured transmitted wave c_4 at the hydrophone receiver. This is done by computing the solution (12) forwardly.

Theoretically, the linear system (12) has a unique solution. But computationally, it is a singular system as its condition number is about 10^{60} for practical data. To produce accurate solutions, we have to rescale the constants C_i or c_i against each other, because the solution data in bone are about 10^{-10} of that from water. In our computation, we use the following variable substitution in (12)

$$C_i = \frac{\bar{C}_i}{\max\{|\lambda+Q|, |Q+R|, |R|, |Q|\}}, \quad i = 1, ..., 4.$$

Here the scaling factor is about 10^{-10}.

In the numerical computation, in addition to the constants listed in Table 1, we used also the following data, $\hat{f} = c_0^2$, $x_s = -1$ and $L = 0.05$. We experimented with different waves ω. In Figure 2, we plotted the real part of constant c_4 (the imaginary part of c_4 is neglected as it is about one-tenth of the real part) in the transmitted wave against the porosity of cancellous bones β for $\omega = 10^5$.

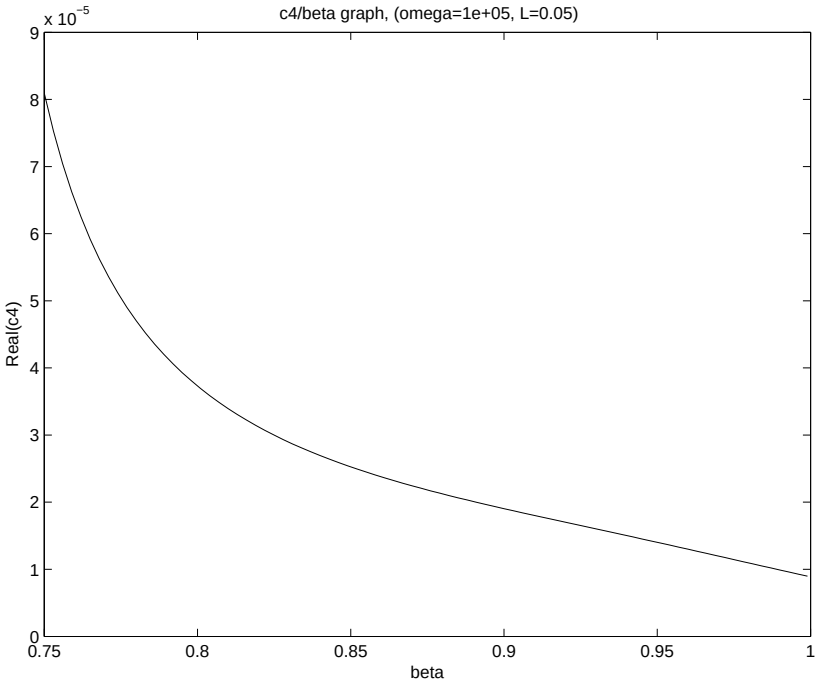


Figure 2. Plot of solution c_4 for different porosity β, $w = 10^5$.

The dependence of c_4 on the porosity β factor for $\omega = 100,000$ in Figure 2, i.e., the average derivative (for $\beta \in [0.75, 1]$), is

$$\frac{dRe(c_4)}{d\beta} = -2.9 \times 10^{-4} = -0.00029.$$

In other words, for frequency $\omega = 10^5$, if the transmitted wave can be measured to 10^{-5} then we would have about two correct digits being recovered for β. To avoid of the dependence on the strength of incident wave, we can use relative measurement. The above averaged derivative indicates that, after divided by the magnitude of c_4, for a relative error of 1% in measured data c_4, the recovered β would have a relative error of only 0.12%! With lower frequencies, we can get

smaller derivative $\frac{dRe(c_4)}{d\beta}$, for example, see Figure 3, where

$$\frac{dRe(c_4)}{d\beta} = 4.6 \times 10^{-4} = 0.00046.$$

Nevertheless, in this case, the relative error in β would be 1.13% if the measured error in c_4 is 1%

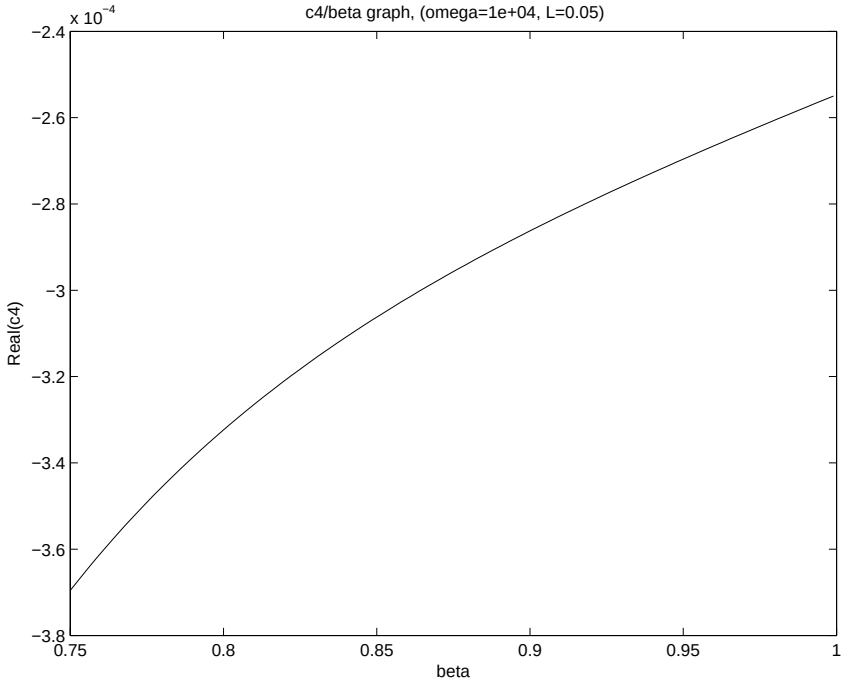


Figure 3. Plot of solution c_4 for different porosity β, $w = 10^4$.

Finally, we listed the relative errors in β against the measured errors in the transmitted wave, the real part of c_4 in (11), for various sound waves. In order to get more accurate averages, we chose the average range of β from 0.81 to 0.96, where the c_4–β curves are monotonic. Numerically, it seems the recovery is more accurate when higher frequency ultrasonic testing wave is used. However, numerically it is more challenging to solve the problem at higher frequencies as the linear system for the coupling (12) is very ill-posed

Frequency ω	Rel. error in β	Average $dRe(c_4)/d\beta$
$1,000$	1.6995%	0.028650
$3,000$	0.6431%	-0.008460
$6,000$	0.6523%	0.002091
$10,000$	1.2354%	0.000399
$30,000$	0.7887%	-0.000085
$60,000$	0.2035%	-0.003949
$100,000$	0.1711%	-0.000139
$300,000$	0.0510%	-0.000040
$600,000$	0.0360%	0.000008
$1,000,000$	0.0299%	-0.000170

Table 2. Relative errors in recovering β at measured relative error 1%.

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Snapshots in Applied Complex Analysis

Proceedings of the Workshop on Recent Trends in Applied Complex Analysis
June 1-5, 2004

Middle East Technical University
Ankara, Turkey

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Preface

The intension of this workshop was as well to instruct newcomers in the field of applied complex analysis about recent trends as to give them the opportunity to meet some of the experts. Among the younger participants were Algerians, Germans, Chinese, Turkish, Tajik, and Vietnamese. The experts came from Armenia, Belarus, China, Germany, France/Poland, Georgian, India, Italy, Russia, Tanzania, Turkey, USA. The workshop was financially supported by the Middle East Technical University in Ankara, the Scientific and Technical Research Council of Turkey and the Ankara Branch of the Turkish Mathematical Society. Their generous support is gratefully acknowledged. Without it the character of the workshop would have been completely different as most of the young generation and many experts in particular from the former SU and countries with weak economy would not have participated.

At the workshop some series of lectures and about 25 talks were given and the selection of manuscript in this volume reflects the range of topics but not the familiar and stimulating atmosphere at the meeting. The comfortable hosting at the guest house and dormitories of the METU, the kind care taking of the local organizers and their helpers and the homogeneous composition of the 30 participants had created a friendly background for an exciting workshop.

The topics vary from complex ordinary differential equations and regular holomorphic systems to complex partial differential equations, from boundary value problems of Riemann, Haseman and Riemann-Hilbert type to Hele Shaw flows and porosity of cancellous bones, from domain perturbation problems to optimization of fixed point methods, from quaternionic to Clifford analysis.

Growth estimations in a certain angular domain for entire finite order solutions to inhomogeneous linear complex ordinary differential equations with entire coefficients are provided under wider conditions as in the past. Motivated by investigations of the geometry of a-points of a meromorphic function the closedness of a- and b-points of an arbitrary polynomial for different a and b are measured and the sum of these measures for several pairs of different points estimated from above where the upper bound is independent of the number of pairs. Combining methods from singularity theory, complex analysis, and ordinary differential equations holonomic Picard-Fuchs systems of differential equations regular singular along hypersurfaces are studied. As an introduction to the theory of boundary value problems for complex partial differential equations of arbitrary order the basic problems of Schwarz, Dirichlet, and Neumann are solved for the inhomogeneous Cauchy-Riemann equation. The Schwarz problem is considered in Wiener domains for some particular quasilinear Bitsadze equation.

Eliptic systems of 2n second order equations with constant leading coefficients are treated in the upper half plane on the basis of the Bitsadze representation and the results applied to anisotropic plane elasticity. Three versions of the modified doubly-periodic second fundamental complete plane strain problem with relative displacements are formulated and solved via the Sherman transform the kernel function of which is given by the Weierstraß ζ-function.

More advanced boundary value problems in complex analysis are the Riemann-Hilbert and the Riemann problem, the first being a generalization of the Schwarz problem and in particular situations related to the latter. The Riemann problem is a special form as well of the general $\mathbb{R}$-linear conjugation problem as of the Haseman problem where also boundary shifts are involved. An $\mathbb{R}$-linear conjugation problem for elliptic functions in a disc is solved in form of Eisenstein series via a transformation to a functional equation. The result is applied to calculate the effective conductivity of a composite material with circular inclusions. A general formulation of the Riemann-Hilbert problem on almost complex manifolds is given within the theory of geometric and topological aspects of holomorphic curves in a loop spaces. Reduction to related problems for analytic functions is used to treat the Haseman boundary value problem for bianalytic functions.

The celebrated Hele-Shaw moving boundary value problem is considered in a doubly connected domain in decomposing it in a Schwarz boundary value problem and an abstract Cauchy-Kowalewski problem. Local existence and uniqueness of the classical solution is proved. The dependence of the porosity of cancellous bones on the measured ultrasonic wave is numerically computed. Here the Biot system of compressional wave equations for a dispersive dissipative fluid-saturated porous medium in the time domain is applied.

With functional analytic and potential theoretic methods domain perturbation problems for boundary value problems for homogeneous elliptic equations are investigated. In particular the Dirichlet and the Neumann problems for the Laplace equation are considered in the n-dimensional Euclidean space. As is well-known the solution of the Picard-Lindelöf integral equation is locally uniquely solvable. This situation often occurs when initial or boundary value problems are solved iteratively. Here the problem is studied how this "local" can be optimized.

Complex analytic methods are not restricted to plane problems. Quaternionic, octonionic, and Clifford analysis are theories efficiently used in mathematical physics where complex analytic methods are applied. Recently higher order Helmholtz operators and their factors were considered in quaternionic analysis as were higher order Laplace operators and their factors in Clifford analysis. On the basis of the quaternionic Stokes formula orthogonal decompositions of the Lebesgue space of square integrable quaternionic-valued functions are given with respect to the poly-α-hyperholomorphic and the polymetaharmonic functions. Higher order Cauchy-Pompeiu representations in Clifford analysis are known for powers of the Dirac operator D and the Laplace operator $\Delta = D\bar{D}$. Here such integral representations are developed for arbitrary operators $D^m\bar{D}^n$ for $m, n \in \mathbb{N}$. Also an addendum to those representation formulas in a universal Clifford algebra in case of an even-dimensional Eulidean space of dimension n is given with respect to D^k for $n \leq k$. Higher order Cauchy-Pompeiu representations for functions of $n, 1 \leq n$, complex variables are available in particular for polydomains. A general formula is given in connection with the operators $\partial_{z_\kappa}^{k_\kappa}\partial_{\bar{z}_\lambda}^{l_\lambda}, k_\kappa, l_\lambda \in \mathbb{N}, 1 \leq \kappa, \lambda \leq n$. Properties of the polyharmonic Green function for the upper half of the complex plane are studied.

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Perturbation problems in potential theory, a functional analytic approach

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Abstract

We consider an approach based on functional analysis and potential theory to investigate domain perturbation problems. In particular, we consider the Dirichlet and Neumann boundary value problems for harmonic functions.

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1 Perturbation problems in potential theory: the Dirichlet problem

In this paper, we plan to consider an approach based on Functional Analysis and Potential Theory to study domain perturbation problems for boundary value problems for homogeneous elliptic equations. For simplicity, we confine our attention to the Laplace equation. In this section, we consider the Dirichlet problem, and in Section 2 we consider the Neumann problem.

We consider the Dirichlet problem for the Laplace equation in a domain of $\mathbb{R}^n$ and ask questions such as that of the dependence of solutions if both the data and the domain are perturbed. Since the dependence of solutions is linear in the boundary data, the question has some interest mainly for the dependence on the domain perturbation. Problems of this type have long been investigated. Here we mention Keldysh [13], Calderón [5], Coifman, McIntosh and Meyer [6], Coifman and Y. Meyer [7], David [8], Verchota (see Meyer [24]), Sokolowski and Zolesio [35], Henry [12] for regular perturbation problems, and Kozlov, Maz'ya and Movchan [14], Maz'ya, Nazarov and Plamenewskii [23], Movchan [26], Ozawa [27], Ward and Keller [38] for singular perturbations. The approach we present is based on Potential Theory and Functional Analysis and exploits the integral equation method for boundary value problems. The idea of exploiting such method in domain perturbation problems is not new. Recentely, it has been applied to study scattering problems by Potthost [29], [30], [28], Potthost and Stratis [31], who have based their results on Fréchet differentiability theorems for various potentials upon small variations of the support (normally a hypersurface). We are mostly interested in results of real analytic dependence for perturbations of arbitrary size and under 'optimal' regularity conditions in Schauder spaces. Our work stems from that of [16], [19], Lanza and Preciso [20], Lanza and Rogosin [21], Preciso and Rogosin [33] for two dimensional problems and of Lanza and Rossi [22] for n-dimensional problems. In particular, this paper exploits heavily results of Lanza and Rossi [22].

We take the domain for our boundary value problem in a special form. We consider a domain whose boundary is of 'sphere-type'. Namely, we assume that the boundary of the domain is of the form $\phi(\partial\mathbb{B}_n)$, where $\mathbb{B}_n$ denotes the unit ball $\{x \in \mathbb{R}^n : |x| < 1\}$ and $\partial\mathbb{B}_n$ denotes the boundary of $\mathbb{B}_n$, and where ϕ is a diffeomorphism of $\partial\mathbb{B}_n$

into $\mathbb{R}^n$ and belongs to the class

$$\mathcal{A}_{\partial\mathbb{B}_n} \equiv \Big\{ \phi \in C^1(\partial\mathbb{B}_n, \mathbb{R}^n) : \phi \text{ is injective}, \\ d\phi(y) \text{ is injective for all } y \in \partial\mathbb{B}_n \Big\}.$$

By the Jordan Leray separation Theorem, $\mathbb{R}^n \setminus \phi(\partial\mathbb{B}_n)$ has exactly two open connected components for all $\phi \in \mathcal{A}_{\partial\mathbb{B}_n}$, and we take our domain to be the bounded open connected component $\mathbb{I}[\phi]$ of $\mathbb{R}^n \setminus \phi(\partial\mathbb{B}_n)$, and we consider the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{I}[\phi], \\ u = g \circ \phi^{(-1)} & \text{on } \phi(\partial\mathbb{B}_n), \end{cases} \tag{1}$$

where g is a given function of $\partial\mathbb{B}_n$ to $\mathbb{R}$, and we are interested into the dependence of the solution of (1) upon ϕ, g. By classical Potential Theory, we can write under reasonable regularity conditions the unique solution $u[\phi, g]$ of (1) in the form

$$u[\phi, g](t) = \int_{\phi(\partial\mathbb{B}_n)} \theta \circ \phi^{(-1)}(s) \frac{\partial}{\partial \nu_\phi(s)} \left[S_n(t-s) \right] d\sigma_s \qquad \forall t \in \mathbb{I}[\phi], \tag{2}$$

where ν_ϕ is the outward unit normal field to $\mathbb{I}[\phi]$ on $\partial\mathbb{I}[\phi] = \phi(\partial\mathbb{B}_n)$, S_n is the fundamental solution of the Laplace operator, *i.e.*, the function of $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_n(\xi) \equiv \begin{cases} \frac{1}{s_n} \log|\xi| & \forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n = 2, \\ \frac{1}{(2-n)s_n} |\xi|^{2-n} & \forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n > 2, \end{cases}$$

where s_n denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n$, and where θ is the unique solution of the Fredholm equation

$$g \circ \phi^{(-1)}(t) = \frac{1}{2} \theta \circ \phi^{(-1)}(t) \\ + \int_{\phi(\partial\mathbb{B}_n)} \theta \circ \phi^{(-1)}(s) \frac{\partial}{\partial \nu_\phi(s)} \left[S_n(t-s) \right] d\sigma_s \quad \forall t \in \phi(\partial\mathbb{B}_n). \tag{3}$$

Thus we plan to study the dependence of $u[\phi, g]$ upon (ϕ, g) by studying the dependence of the solution θ of (3) upon (ϕ, g) and the dependence of the right-hand side of (2) upon ϕ, θ. Now we note that equation (3) is defined on the ϕ-dependent domain $\phi(\partial\mathbb{B}_n)$. However,

we prefer to work with a fixed domain. Then we change the variables in (3) and we obtain

$$g(x) = \frac{1}{2}\theta(x) + \int_{\phi(\partial\mathbb{B}_n)} \theta\circ\phi^{(-1)}(s)\frac{\partial}{\partial\nu_\phi(s)}\left[S_n(\phi(x)-s)\right]\,d\sigma_s \quad \forall x \in \partial\mathbb{B}_n\,. \tag{4}$$

We wish to recast (4) into an abstract equation in Banach space. Thus we now introduce some notation. We denote by $\mathbb{N}$ the set of natural numbers including 0. Then for all $m \in \mathbb{N}$, $\alpha \in]0,1[$, and bounded open subsets Ω of $\mathbb{R}^n$, we denote by $C^m(\mathrm{cl}\Omega)$ the space of m times continuously differentiable functions of the closure $\mathrm{cl}\Omega$ of Ω to $\mathbb{R}$, and by $C^{m,\alpha}(\mathrm{cl}\Omega)$ the Schauder linear subspace of $C^m(\mathrm{cl}\Omega)$ of those functions which have α-Hölder continuous m-th order derivatives in $\mathrm{cl}\Omega$. Then $C^{m,\alpha}(\partial\Omega)$ is defined as the space of traces of functions of $C^{m,\alpha}(\mathrm{cl}\Omega)$ onto $\partial\Omega$. We think of $C^m(\mathrm{cl}\Omega)$ as endowed with the norm of uniform convergence in $\mathrm{cl}\Omega$ of all derivatives up to order m. We think of the Schauder spaces $C^{m,\alpha}(\mathrm{cl}\Omega)$, $C^{m,\alpha}(\partial\Omega)$ as endowed with their usual norm. For standard properties of functions in Schauder spaces, we refer the reader to Gilbarg and Trudinger [10] (see also [15, §2, Lem. 3.1, 4.26, Thm. 4.28], Lanza and Rossi [22, §2].) Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $C^{m,\alpha}(\mathrm{cl}\Omega, \mathbb{D})$ denotes $\{f \in (C^{m,\alpha}(\mathrm{cl}\Omega))^n : f(\mathrm{cl}\Omega) \subseteq \mathbb{D}\}$. Also, we note that $\mathcal{A}_{\partial\mathbb{B}_n}$ is open in $C^1(\partial\mathbb{B}_n, \mathbb{R}^n)$ (see [15, Prop. 4.29], Lanza and Rossi [22, Lem. 2.5].)

We now define the map Λ of $(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m,\alpha}(\partial\mathbb{B}_n)^2$ to $C^{m,\alpha}(\partial\mathbb{B}_n)$ by setting

$$\Lambda[\phi, g, \theta](\cdot) \equiv -g(\cdot) + \frac{1}{2}\theta(\cdot) + \int_{\phi(\partial\mathbb{B}_n)} \theta\circ\phi^{(-1)}(s)\frac{\partial}{\partial\nu_\phi(s)}\left[S_n(\phi(\cdot)-s)\right]\,d\sigma_s \tag{5}$$

for all $(\phi, g, \theta) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m,\alpha}(\partial\mathbb{B}_n)^2$. Then we can consider equation (4) in the abstract form

$$\Lambda[\phi, g, \theta] = 0 \tag{6}$$

and we plan to analyze equation (6) by means of the Implicit Function Theorem. To do so, we must understand the regularity of the nonlinear operator Λ, and thus we resort to the following Theorem of Lanza and Rossi [22, Thm. 3.12].

Theorem 1 *Let $m \in \mathbb{N}\backslash\{0\}$, $\alpha \in]0,1[$. The map $W[\cdot,\cdot]$ of $(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m,\alpha}(\partial\mathbb{B}_n)$ to $C^{m,\alpha}(\partial\mathbb{B}_n)$ defined by*

$$W[\phi, f](x) \equiv \int_{\phi(\partial\mathbb{B}_n)} \frac{\partial}{\partial\nu_\phi(s)}\left[S_n(\phi(x)-s)\right] f\circ\phi^{(-1)}(s)\,d\sigma_s \quad \forall x \in \partial\mathbb{B}_n\,,$$

for all $(\phi, f) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m,\alpha}(\partial\mathbb{B}_n)$ *is real analytic.*

For the definition and properties of real analytic operators in Banach spaces, we refer the reader to Prodi and Ambrosetti [34]. Now let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. For a fixed $\phi \in C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}$, the set $\mathbb{I}[\phi]$ is a manifold with boundary of class $C^{m,\alpha}$ imbedded in $\mathbb{R}^n$ (cf. *e.g.*, Lanza and Rossi [22, Lem. 2.6]), and thus classical Potential Theory ensures that for each $g \in C^{m,\alpha}(\partial\mathbb{B}_n)$, equation (4) admits a unique solution $\theta \in C^{m,\alpha}(\partial\mathbb{B}_n)$, which we denote by $\hat{\theta}[\phi, g]$. For the existence and uniqueness of a continuous solution θ, we refer to Folland [9, Ch. 3], and for its regularity we refer to Theorem 14 (i) of Appendix C of this paper. We plan to analyze the regularity of $\hat{\theta}[\cdot,\cdot]$ by applying the Implicit Function Theorem to equation (6). Then by Theorem 1 and by classical Potential Theory, we introduce the following Theorem on Λ.

Theorem 2 *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. *The operator* Λ *is real analytic, and for each zero* (ϕ_0, g_0, θ_0) *of* Λ, *the partial differential* $\partial_\theta \Lambda[\phi_0, g_0, \theta_0]$ *of* Λ *with respect to the variable* θ *at the point* (ϕ_0, g_0, θ_0) *is a linear homeomorphism of* $C^{m,\alpha}(\partial\mathbb{B}_n)$ *onto itself.*

Then by applying the Implicit Function Theorem to equation (6) around a point $(\phi_0, g_0, \hat{\theta}[\phi_0, g_0])$ (cf. *e.g.*, Prodi and Ambrosetti [34, Thm. 11.6, p. 101]), we deduce the validity of the following.

Theorem 3 *Let* $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. *The map* $\hat{\theta}[\cdot,\cdot]$ *of* $(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m,\alpha}(\partial\mathbb{B}_n)$ *to* $C^{m,\alpha}(\partial\mathbb{B}_n)$ *which takes a pair* (ϕ, g) *to* $\hat{\theta}[\phi, g]$, *is real analytic.*

We now study the dependence of $u[\phi, g]$ upon (ϕ, g) by exploiting formula (2) and Theorem 3. To do so, we wish to change the variables in formula (2) by means of the function ϕ. We note that for each $\phi \in C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}$ with $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$, there exists a unique $\tilde{\sigma}[\phi] \in C^{m-1,\alpha}(\partial\mathbb{B}_n)$ such that $\tilde{\sigma}[\phi] > 0$ and

$$\int_{\phi(\partial\mathbb{B}_n)} \omega(s)\, d\sigma_s = \int_{\partial\mathbb{B}_n} \omega \circ \phi(y) \tilde{\sigma}[\phi](y)\, d\sigma_y \qquad \forall \omega \in L^1(\phi(\partial\mathbb{B}_n)),$$

(cf. *e.g.*, Lanza and Rossi [22, p. 166].) Then we note that we can rewrite formula (2) in the form

$$u[\phi, g](t) = -\int_{\partial\mathbb{B}_n} \hat{\theta}[\phi, g](y)(\nu_\phi \circ \phi(y)) DS_n(t - \phi(y)) \tilde{\sigma}[\phi](y)\, d\sigma_y \ \ \forall t \in \mathbb{I}[\phi], \tag{7}$$

where DS_n denotes the gradient matrix of S_n. We now wish to show that $u[\phi, g](t)$ depends real analytically on (ϕ, g). Then we need the following technical statement.

Proposition 1 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$.*

(i) The map $\tilde{\sigma}[\cdot]$ of $C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}$ to $C^{m-1,\alpha}(\partial\mathbb{B}_n)$ which takes ϕ to $\tilde{\sigma}[\phi]$ is real analytic.

(ii) The map of $C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}$ to $C^{m-1,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n)$ which takes ϕ to $\nu_\phi \circ \phi$ is real analytic.

(iii) Let F be a real analytic map of $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$. Let Ω be a bounded open subset of $\mathbb{R}^n$. Then the map H of $\{(\phi, f) \in C^0(\partial\mathbb{B}_n, \mathbb{R}^n) \times L^1(\partial\mathbb{B}_n) : \phi(\partial\mathbb{B}_n) \cap \mathrm{cl}\Omega = \emptyset\}$ to $C^0(\mathrm{cl}\Omega)$ which takes (ϕ, f) to the function $H[\phi, f]$ of $\mathrm{cl}\Omega$ to $\mathbb{R}$ defined by

$$H[\phi, f](t) \equiv \int_{\partial\mathbb{B}_n} F\left(t - \phi(y)\right) f(y)\, d\sigma_y \qquad \forall t \in \mathrm{cl}\Omega\,,$$

is real analytic.

Proof For statements (i), (ii), see Lanza and Rossi [22, Prop. 3.13]. We now prove statement (iii). Let $\mathrm{id}_{\mathrm{cl}\Omega}$ denote the identity map in $\mathrm{cl}\Omega$. The map of $\{\phi \in C^0(\partial\mathbb{B}_n, \mathbb{R}^n) : \phi(\partial\mathbb{B}_n) \cap \mathrm{cl}\Omega = \emptyset\}$ to $C^0(\mathrm{cl}\Omega \times \partial\mathbb{B}_n, \mathbb{R}^n \setminus \{0\})$ which takes ϕ to the function $\mathrm{id}_{\mathrm{cl}\Omega}(t) - \phi(y)$ of the variable $(t, y) \in \mathrm{cl}\Omega \times \partial\mathbb{B}_n$ is real analytic. The map of $C^0(\mathrm{cl}\Omega \times \partial\mathbb{B}_n, \mathbb{R}^n \setminus \{0\})$ to $C^0(\mathrm{cl}\Omega \times \partial\mathbb{B}_n)$ which takes a function Φ to its composite $F \circ \Phi$ is real analytic (cf. Böhme and Tomi [2, p. 10], Henry [12, p. 29], Valent [37, Thm. 5.2, p. 44], who considered the more elaborate case of the Schauder spaces $C^{m,\alpha}$.) Then to complete the proof we just need to observe that the map which takes a pair of functions (g, f) of $C^0(\mathrm{cl}\Omega \times \partial\mathbb{B}_n) \times L^1(\partial\mathbb{B}_n)$ to $\int_{\partial\mathbb{B}_n} g(\cdot, y) f(y)\, d\sigma_y$ in $C^0(\mathrm{cl}\Omega)$ is real analytic. □

Then by formula (7), by Theorem 3 and by Proposition 1, we can immediately deduce the validity of the following Theorem.

Theorem 4 *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let $(\phi_0, g_0) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m,\alpha}(\partial\mathbb{B}_n)$. Let Ω be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega \subseteq \mathbb{I}[\phi_0]$. Then there exists an open neighborhood $\mathcal{U}$ of (ϕ_0, g_0) in $(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m,\alpha}(\partial\mathbb{B}_n)$ such that*

(i) $\mathrm{cl}\Omega \subseteq \mathbb{I}[\phi]$ for all $(\phi, g) \in \mathcal{U}$.

(ii) The operator U_Ω *of* $\mathcal{U}$ *to* $C^0(\mathrm{cl}\Omega)$ *which takes* (ϕ, g) *to the restriction* $u[\phi, g]_{|\mathrm{cl}\Omega}$ *of* $u[\phi, g]$ *to* $\mathrm{cl}\Omega$ *is real analytic.*

We note that in case $n = 2$, Theorem 4 is contained in Preciso and Rogosin [33, Prop. 4.1], who proved a corresponding statement for the Schwarz operator, by exploiting a result on the dependence of the conformal representation upon perturbation of the domain. We also note that by exploiting an argument of Potthast [29], [30], one could prove a variant of Theorem 4, case $m = 1$, by replacing the regularity assumption ϕ of class $C^{1,\alpha}$ with the stronger regularity assumption ϕ of class C^2, and by replacing the real analyticity by Fréchet differentiability.

We now consider the map U of

$$\mathcal{F}^{m,\alpha} \equiv \{(\phi, g, t) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m,\alpha}(\partial\mathbb{B}_n) \times \mathbb{R}^n : t \in \mathbb{I}[\phi]\}$$

to $\mathbb{R}$ defined by

$$U[\phi, g, t] \equiv u[\phi, g](t) \qquad \forall (\phi, g, t) \in \mathcal{F}^{m,\alpha}\,, \tag{8}$$

and we ask whether U is real analytic. By Theorem 4 we know that for each $(\phi_0, g_0, t_0) \in \mathcal{F}^{m,\alpha}$, the map $U[\phi, g, t_0]$ is real analytic in the variable (ϕ, g) around the point (ϕ_0, g_0). Moreover, by standard properties of harmonic functions, we know that $U[\phi_0, g_0, t]$ is real analytic in the variable t around the point t_0. However, such facts do not imply the real analyticity of U in the variable (ϕ, g, t) (cf. Boman [3].) To prove the real analyticity of U, we introduce the following technical statement on composition operators, which we prove in Appendix A.

Proposition 2 *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$*. Then the map* Ξ *of the subspace*

$$C^0_h(\mathrm{cl}\Omega) \equiv \left\{u \in C^0(\mathrm{cl}\Omega) \cap C^2(\Omega) : \Delta u(t) = 0 \ \forall t \in \Omega\right\} \tag{9}$$

of $C^0(\mathrm{cl}\Omega)$ *endowed with the norm of the uniform convergence in* $\mathrm{cl}\Omega$ *to* $\mathbb{R}$ *defined by*

$$\Xi[u, t] = u(t) \qquad \forall (u, t) \in C^0_h(\mathrm{cl}\Omega) \times \Omega\,,$$

is real analytic.

By combining Theorem 4 and Proposition 2, we deduce the validity of the following.

Theorem 5 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. The map U of $\mathcal{F}^{m,\alpha}$ to $\mathbb{R}$ defined by (8) is real analytic.*

We now turn to consider the energy integral of the solution $u[\phi, g]$, *i.e.*, the integral

$$\mathcal{E}[\phi, g] \equiv \int_{\mathbb{I}[\phi]} |\nabla u[\phi, g](t)|^2 \, dt \,. \tag{10}$$

By the Divergence Theorem, we have

$$\mathcal{E}[\phi, g] = \int_{\partial \mathbb{B}_n} g \left\{ \left(\frac{\partial}{\partial \nu_\phi} u[\phi, g] \right) \circ \phi \right\} \tilde{\sigma}[\phi] \, d\sigma \,. \tag{11}$$

Thus we must understand how $\left(\frac{\partial}{\partial \nu_\phi} u[\phi, g]\right) \circ \phi$ depends on ϕ and g. To do so, we introduce some notation and resuts of Lanza and Rossi [22]. For each bounded open connected subset Ω of $\mathbb{R}^n$ of class C^1, we set

$$\mathcal{A}_{\mathrm{cl}\,\Omega} \equiv \left\{ \Phi \in C^1(\mathrm{cl}\,\Omega, \mathbb{R}^n) : \Phi \text{ is injective, } \det D\Phi(x) \neq 0, \ \forall x \in \mathrm{cl}\,\Omega \right\} .$$

The set $\mathcal{A}_{\mathrm{cl}\,\Omega}$ is known to be open in $C^1(\mathrm{cl}\,\Omega, \mathbb{R}^n)$ (cf. [15, Cor. 4.24, Prop. 4.29].) If $\delta > 0$, we set

$$\begin{aligned} \mathbb{A}_\delta &\equiv \{x \in \mathbb{R}^n : 1 - \delta < |x| < 1 + \delta\} \,, \\ \mathbb{A}_\delta^+ &\equiv \{x \in \mathbb{R}^n : 1 - \delta < |x| < 1\} \,, \\ \mathbb{A}_\delta^- &\equiv \{x \in \mathbb{R}^n : 1 < |x| < 1 + \delta\} \,, \end{aligned}$$

Then we have the following (cf. Lanza and Rossi [22, Prop. 2.8].)

Proposition 3 *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1]$. If ϕ_0 belongs to $C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}$, then there exist $\delta \in]0,1[$, an open neighborhood $\mathcal{U}_0$ of ϕ_0 contained in $C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}$, and a real analytic extension operator $\mathbf{E}_0$ of $\mathcal{U}_0$ to $C^{m,\alpha}(\mathrm{cl}\mathbb{A}_\delta, \mathbb{R}^n) \cap \mathcal{A}_{\mathrm{cl}\mathbb{A}_\delta}$ such that*

(i) $\mathbf{E}_0[\phi]_{|\partial\mathbb{B}_n} = \phi$, *for all* $\phi \in \mathcal{U}_0$.

(ii) $\mathbf{E}_0[\phi]\left(\mathbb{A}_\delta^+\right)$ *is a bounded open connected subset of* $\mathbb{R}^n$ *of class* $C^{m,\alpha}$ *contained in* $\mathbb{I}[\phi]$, $\mathbf{E}_0[\phi]\left(\mathbb{A}_\delta^-\right)$ *is a bounded open connected subset of* $\mathbb{R}^n$ *of class* $C^{m,\alpha}$ *contained in* $\mathbb{R}^n \setminus \mathrm{cl}\mathbb{I}[\phi]$, *for all* $\phi \in \mathcal{U}_0$.

The set

$$\mathcal{A}'_{\mathrm{cl}\mathbb{A}_\delta} \equiv \left\{\Phi \in \mathcal{A}_{\mathrm{cl}\mathbb{A}_\delta} : \Phi(\mathbb{A}^+_\delta) \subseteq \mathbb{I}[\Phi_{|\partial\mathbb{B}_n}]\right\}$$

is open in $\mathcal{A}_{\mathrm{cl}\mathbb{A}_\delta}$ and $\mathbf{E}_0$ maps $\mathcal{U}_0$ to $\mathcal{A}'_{\mathrm{cl}\mathbb{A}_\delta}$ (cf. Lanza and Rossi [22, Lem. 2.9].) We now introduce some notation on double layer potentials. Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. We set

$$w^+[\phi, f](t) \equiv \int_{\phi(\partial\mathbb{B}_n)} f \circ \phi^{(-1)}(s) \frac{\partial}{\partial \nu_\phi(s)} \left[S_n(t-s)\right] d\sigma_s \quad \forall t \in \mathbb{I}[\phi]\,, \tag{12}$$

for each $\phi \in C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}$, and $f \in C^{m,\alpha}(\partial\mathbb{B}_n)$. Then by classical Potential Theory, we know that $w^+[\phi, f]$ can be extended with continuity to $\mathrm{cl}\mathbb{I}[\phi]$. We denote the corresponding extension with the same symbol. Then we have the following statement which has been proved in Lanza and Rossi [22, Prop. 3.11].

Proposition 4 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$, $\delta \in]0,1[$. Let $W^+[\Phi, f]$ denote the continuous extension to $\mathrm{cl}\mathbb{A}^+_\delta$ of the function $w^+[\Phi_{|\partial\mathbb{B}_n}, f] \circ \Phi_{|\mathbb{A}^+_\delta}$, for all $(\Phi, f) \in (C^{m,\alpha}(\mathrm{cl}\mathbb{A}_\delta, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\mathbb{A}_\delta}) \times C^{m,\alpha}(\partial\mathbb{B}_n)$. Then the map $W^+[\cdot,\cdot]$ of the set $\left(C^{m,\alpha}(\mathrm{cl}\mathbb{A}_\delta, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\mathbb{A}_\delta}\right) \times C^{m,\alpha}(\partial\mathbb{B}_n)$ to $C^{m,\alpha}\left(\mathrm{cl}\mathbb{A}^+_\delta\right)$ which takes (Φ, f) to $W^+[\Phi, f]$ is real analytic.*

Now we have the following elementary Lemma.

Lemma 1 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha, \delta \in]0,1[$. The map of $(C^{m,\alpha}(\mathrm{cl}\mathbb{A}_\delta, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\mathbb{A}_\delta}) \times C^{m,\alpha}(\mathrm{cl}\mathbb{A}^+_\delta)$ to $C^{m-1,\alpha}(\partial\mathbb{B}_n)$ which takes a pair (Φ, G) to $\frac{\partial}{\partial\nu_\phi}(G \circ \Phi^{(-1)}{}_{|\Phi(\mathrm{cl}\mathbb{A}^+_\delta)}) \circ \phi$, with $\phi \equiv \Phi_{|\partial\mathbb{B}_n}$ is real analytic.*

The same statement applies if $\mathbb{A}^+_\delta$ is replaced by $\mathbb{A}^-_\delta$.

Proof It suffices to note that

$$\frac{\partial}{\partial\nu_\phi}(G \circ \Phi^{(-1)}) \circ \phi = \left(\left(D(G \circ \Phi^{(-1)})\right) \circ \phi\right) \nu_\phi \circ \phi = (DG)(D\Phi)^{-1}\nu_\phi \circ \phi$$

and to invoke Proposition 1 (ii). □

Then we have the following.

Proposition 5 *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. The map of $(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m,\alpha}(\partial\mathbb{B}_n)$ to $C^{m-1,\alpha}(\partial\mathbb{B}_n)$ which takes a pair (ϕ, g) to $(\frac{\partial}{\partial\nu_\phi} u[\phi, g]) \circ \phi$ is real analytic*

Proof Let $(\phi_0, g_0) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m,\alpha}(\partial\mathbb{B}_n)$. Let $\mathbf{E}_0$, $\mathcal{W}_0$, $\delta > 0$ be as in Proposition 3 for ϕ_0. By Lemma 1 and by Proposition 3, it suffices to show that the map of $\mathcal{W}_0 \times C^{m,\alpha}(\partial\mathbb{B}_n)$ to $C^{m,\alpha}(\mathrm{cl}\mathbb{A}_\delta^+)$ which takes (ϕ, g) to $u[\phi, g] \circ \mathbf{E}_0[\phi]_{|\mathrm{cl}\mathbb{A}_\delta^+}$ is real analytic. By formula (7), we have $u[\phi, g] \circ \mathbf{E}_0[\phi]_{|\mathrm{cl}\mathbb{A}_\delta^+} = W^+[\mathbf{E}_0[\phi], \hat{\theta}[\phi, g]]$. Thus the conclusion follows by Propositions 3, 4 and by Theorem 3. □

Then by formula (11), and Proposition 1 (i), we immediately deduce the validity of the following.

Theorem 6 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1[$. The functional $\mathcal{E}$ of $(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m,\alpha}(\partial\mathbb{B}_n)$ to $\mathbb{R}$ defined by (10) for all $(\phi, g) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m,\alpha}(\partial\mathbb{B}_n)$ is real analytic.*

2 Perturbation problems in potential theory: the Neumann problem

We now consider the Neumann boundary value problem

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{I}[\phi]\,, \\ \frac{\partial v}{\partial \nu_\phi} = g \circ \phi^{(-1)} & \text{on } \phi(\partial\mathbb{B}_n)\,, \\ \int_{\phi(\partial\mathbb{B}_n)} v \, d\sigma = 0\,, \end{cases} \tag{13}$$

where g is a given function of $\partial\mathbb{B}_n$ to $\mathbb{R}$, and we are interested into the dependence of the solution of (13) upon ϕ, g. As is well known, for any reasonable choice of regularity conditions for ϕ, g, problem (13) has a unique solution exactly when the compatibility condition

$$\int_{\phi(\partial\mathbb{B}_n)} g \circ \phi^{(-1)} \, d\sigma = 0 \tag{14}$$

holds. Now condition (14) is a nonlinearly ϕ-dependent constraint on the data g. In our analysis, we find convenient to work with no constraints on the data. Thus we choose to replace problem (13) by the problem

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{I}[\phi]\,, \\ \frac{\partial v}{\partial \nu_\phi} = g \circ \phi^{(-1)} - \left\{ \int_{\phi(\partial\mathbb{B}_n)} d\sigma \right\}^{-1} \int_{\phi(\partial\mathbb{B}_n)} g \circ \phi^{(-1)} \, d\sigma & \text{on } \phi(\partial\mathbb{B}_n)\,, \\ \int_{\phi(\partial\mathbb{B}_n)} v \, d\sigma = 0\,, \end{cases} \tag{15}$$

which under reasonable regularity conditions can be solved for all data g, and whose solutions coincide with those of problem (13) if condition (14) is satisfied.

By classical Potential Theory, we can write the solution $v[\phi, g]$ of (15) in the form

$$v[\phi,g](t) = \int_{\phi(\partial\mathbb{B}_n)} \vartheta \circ \phi^{(-1)}(s) S_n(t-s)\, d\sigma_s \tag{16}$$
$$- \left\{ \int_{\phi(\partial\mathbb{B}_n)} d\sigma \right\}^{-1} \int_{\phi(\partial\mathbb{B}_n)} \left\{ \int_{\phi(\partial\mathbb{B}_n)} \vartheta \circ \phi^{(-1)}(s) S_n(t-s)\, d\sigma_s \right\} d\sigma_t \,,$$

for all $t \in \mathbb{I}[\phi]$, where ϑ is the unique solution of the problem

$$\begin{cases} -g \circ \phi^{(-1)}(t) + \left\{ \int_{\phi(\partial\mathbb{B}_n)} d\sigma \right\}^{-1} \int_{\phi(\partial\mathbb{B}_n)} g \circ \phi^{(-1)} d\sigma \\ -\frac{1}{2} \vartheta \circ \phi^{(-1)}(t) \\ + \int_{\phi(\partial\mathbb{B}_n)} \vartheta \circ \phi^{(-1)}(s) \frac{\partial}{\partial \nu_\phi(t)} \left[S_n(t-s) \right] d\sigma_s = 0 \\ \forall t \in \phi(\partial\mathbb{B}_n), \int_{\phi(\partial\mathbb{B}_n)} \vartheta \circ \phi^{(-1)}\, d\sigma = 1 \,. \end{cases} \tag{17}$$

As for the Dirichlet problem, we now recast problem (17) into an abstract form. To do so, we define the map Ψ of $(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)^2$ to $C^{m-1,\alpha}(\partial\mathbb{B}_n) \times \mathbb{R}$ by setting

$$\Psi[\phi, g, \vartheta](x) \equiv \left(\Psi_j[\phi, g, \vartheta](x)\right)_{j=1,2} \tag{18}$$
$$\equiv \Bigg(-g(x) + \left\{ \int_{\partial\mathbb{B}_n} \tilde{\sigma}[\phi]\, d\sigma \right\}^{-1} \int_{\partial\mathbb{B}_n} g \tilde{\sigma}[\phi]\, d\sigma$$
$$-\frac{1}{2}\vartheta(x) + \int_{\partial\mathbb{B}_n} \vartheta(y) (\nu_\phi \circ \phi(x)) DS_n(\phi(x) - \phi(y)) \tilde{\sigma}[\phi](y)\, d\sigma_y,$$
$$\int_{\partial\mathbb{B}_n} \vartheta \tilde{\sigma}[\phi]\, d\sigma - 1 \Bigg) \quad \forall x \in \partial\mathbb{B}_n \,,$$

for all (ϕ, g, ϑ) in the domain of Ψ. Then equation

$$\Psi[\phi, g, \vartheta] = 0 \,, \tag{19}$$

is equivalent to problem (17) for all (ϕ, g, ϑ) in the domain of Ψ. Classical Potential Theory ensures that for each (ϕ, g) in $(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap$

$\mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$ there exists one and only one solution $\vartheta \in C^{m-1,\alpha}(\partial\mathbb{B}_n)$ of (17), or of equation (19), which we denote by $\hat{\vartheta}[\phi, g]$. For the existence and uniqueness of a continuous solution ϑ, we refer to Folland [9, Ch. 3], and for its regularity we refer to Theorem 14 (ii) of Appendix C. We plan to analyze the regularity of $\hat{\vartheta}[\cdot,\cdot]$ by applying the Implicit Function Theorem to equation (19) (although as we shall see a direct application is not possible.) To do so, we must understand the regularity of the nonlinear operator Ψ. Then we introduce some notation about simple layer potentials. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. We set

$$v^+[\phi, f](t) \equiv \int_{\phi(\partial\mathbb{B}_n)} f \circ \phi^{(-1)}(s) S_n(t-s)\, d\sigma_s \quad \forall t \in \mathbb{I}[\phi]\,, \tag{20}$$

for all $(\phi, f) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$. Then by classical Potential Theory, we know that $v^+[\phi, f]$ can be extended with continuity to $\mathrm{cl}\mathbb{I}[\phi]$. We denote the corresponding extension by the same symbol. Then we have the following two statements, which have been proved in Lanza and Rossi [22, Prop. 3.11, Thm. 3.12].

Proposition 6 *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$, $\delta \in]0,1[$. Let $V^+[\Phi, f]$ denote the continuous extension to $\mathrm{cl}\mathbb{A}^+_\delta$ of the function $v^+[\Phi_{|\partial\mathbb{B}_n}, f] \circ \Phi_{|\mathbb{A}^+_\delta}$, for all $(\Phi, f) \in (C^{m,\alpha}(\mathrm{cl}\mathbb{A}_\delta, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\mathbb{A}_\delta}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$. Then the map $V^+[\cdot,\cdot]$ of $\left(C^{m,\alpha}(\mathrm{cl}\mathbb{A}_\delta, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\mathbb{A}_\delta}\right) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$ to $C^{m,\alpha}\left(\mathrm{cl}\mathbb{A}^+_\delta\right)$ which takes (Φ, f) to $V^+[\Phi, f]$ is real analytic.*

Theorem 7 *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. The map $V[\cdot,\cdot]$ of $(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$ to the space $C^{m,\alpha}(\partial\mathbb{B}_n)$ defined by*

$$V[\phi, f](x) \equiv \int_{\phi(\partial\mathbb{B}_n)} f \circ \phi^{(-1)}(s) S_n(\phi(x) - s)\, d\sigma_s \quad \forall x \in \partial\mathbb{B}_n,$$

for all $(\phi, f) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$ is real analytic.

Then we can deduce the validity of the following.

Proposition 7 *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Then the map V_* of the set $(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$ to $C^{m-1,\alpha}(\partial\mathbb{B}_n)$ defined by*

$$V_*[\phi, f](x) \equiv \int_{\partial\mathbb{B}_n} f(y)(\nu_\phi \circ \phi(x)) DS_n(\phi(x) - \phi(y)) \tilde{\sigma}[\phi](y)\, d\sigma_y \ \ \forall x \in \partial\mathbb{B}_n\,,$$

for all $(\phi, f) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$*, is real analytic.*

Proof It clearly suffices to show that if $(\phi_0, f_0) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$, then $V_*[\cdot,\cdot]$ is real analytic in a neighborhood of (ϕ_0, f_0). Now let $\mathcal{W}_0$, δ, $\mathbf{E}_0$ be as in Proposition 3. By classical Potential Theory and by (20), we have

$$-\frac{1}{2}f(t) + V_*[\phi, f](t) = \left\{\frac{\partial}{\partial\nu_\phi(t)}\left(v^+[\phi, f]\right)\right\} \circ \phi(t) \quad \forall t \in \partial\mathbb{B}_n\,,$$

for all $(\phi, f) \in \mathcal{W}_0 \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$. By Lemma 1 and by Proposition 3, it suffices to show that the map of $\mathcal{W}_0 \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$ to $C^{m,\alpha}(\mathrm{cl}\mathbb{A}^+_\delta)$ which takes (ϕ, f) to $v^+[\phi, f] \circ \mathbf{E}_0[\phi]_{|\mathrm{cl}\mathbb{A}^+_\delta} = V^+[\mathbf{E}_0[\phi], f]$ is real analytic. Thus the conclusion follows by Propositions 3, 6. □

Then we have the following statement on Ψ.

Theorem 8 *Let* $m \in \mathbb{N} \setminus \{0\}$*,* $\alpha \in]0, 1[$*. The operator* Ψ *is real analytic and satisfies*

$$\int_{\partial\mathbb{B}_n} \Psi_1[\phi, g, \vartheta]\tilde{\sigma}[\phi]\, d\sigma = 0\,, \tag{21}$$

for all (ϕ, g, ϑ) *in the domain of* Ψ*.*

If $(\phi_0, g_0, \vartheta_0) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)^2$ *is a zero of* Ψ*, then the partial differential* $\partial_\vartheta\Psi[\phi_0, g_0, \vartheta_0]$ *of* Ψ *with respect to the variable* ϑ *at* $(\phi_0, g_0, \vartheta_0)$ *is a linear homeomorphism of* $C^{m-1,\alpha}(\partial\mathbb{B}_n)$ *onto the subspace*

$$V^{m,\alpha}_{\phi_0} \equiv \left\{(\gamma, \omega) \in C^{m-1,\alpha}(\partial\mathbb{B}_n) \times \mathbb{R} : \int_{\partial\mathbb{B}_n} \gamma\tilde{\sigma}[\phi_0]\, d\sigma = 0\right\}$$

of $C^{m-1,\alpha}(\partial\mathbb{B}_n) \times \mathbb{R}$*.*

Proof By Propositions 1 and 7, we deduce that Ψ is real analytic. Equality (21) is an immediate consequence of Fubini Theorem, and of the well known identity

$$\int_{\phi(\partial\mathbb{B}_n)} \frac{\partial}{\partial\nu_\phi(s)}\left[S_n(t-s)\right]\, d\sigma = \frac{1}{2} \qquad \forall t \in \phi(\partial\mathbb{B}_n)\,. \tag{22}$$

We now turn to prove the statement on the linearized problem. By standard calculus in Banach space, we have

$$\begin{aligned}&\partial_\vartheta \Psi[\phi_0, g_0, \vartheta_0](\overline{\vartheta})\\ &= \Big(-\frac{1}{2}\overline{\vartheta}(\cdot) + \int_{\partial\mathbb{B}_n} \overline{\vartheta}(y)(\nu_{\phi_0}\circ\phi_0(\cdot))DS_n(\phi_0(\cdot)-\phi_0(y))\tilde{\sigma}[\phi_0](y)\,d\sigma_y,\\ &\int_{\partial\mathbb{B}_n} \overline{\vartheta}\tilde{\sigma}[\phi_0]\,d\sigma\Big) \quad \forall \overline{\vartheta} \in C^{m-1,\alpha}(\partial\mathbb{B}_n)\,. \end{aligned} \tag{23}$$

We now show that for each $(\gamma,\omega) \in V^{m,\alpha}_{\phi_0}$ there exists one and only one $\overline{\vartheta} \in C^{m-1,\alpha}(\partial\mathbb{B}_n)$ such that

$$\partial_\vartheta \Psi[\phi_0, g_0, \vartheta_0](\overline{\vartheta}) = (\gamma,\omega)\,, \tag{24}$$

which is a system of two equations, one for each component of Ψ. Since the map of $C^{m-1,\alpha}(\partial\mathbb{B}_n)$ to $C^{m-1,\alpha}(\phi_0(\partial\mathbb{B}_n))$ which takes $\overline{\vartheta}$ to $\overline{\vartheta}\circ\phi_0^{(-1)}$ is an isomorphism, classical Potential Theory implies that the space

$$\begin{aligned}&\Big\{\overline{\vartheta} \in C^{m-1,\alpha}(\partial\mathbb{B}_n): \qquad (25)\\ &-\frac{1}{2}\overline{\vartheta}(\cdot) + \int_{\partial\mathbb{B}_n} \overline{\vartheta}(y)(\nu_{\phi_0}\circ\phi_0(\cdot))DS_n(\phi_0(\cdot)-\phi_0(y))\tilde{\sigma}[\phi_0](y)\,d\sigma_y = 0\Big\}\\ &= \Big\{\overline{\vartheta} \in C^{m-1,\alpha}(\partial\mathbb{B}_n):\\ &\quad -\frac{1}{2}\overline{\vartheta}\circ\phi_0^{(-1)}(t) + \int_{\phi_0(\partial\mathbb{B}_n)} \overline{\vartheta}\circ\phi_0^{(-1)}(s)\frac{\partial}{\partial\nu_{\phi_0}(t)}\left[S_n(t-s)\right]\,d\sigma_s = 0\Big\}\end{aligned}$$

is one dimensional, and admits a generator $\overline{\vartheta}_0$ (cf. *e.g.*, Folland [9, Prop. 3.37, p. 181], Theorem 14 (ii) of Appendix C.) If $\int_{\partial\mathbb{B}_n} \overline{\vartheta}_0\tilde{\sigma}[\phi_0]d\sigma = 0$, then the 'transpose' equation

$$-\frac{1}{2}\tau\circ\phi_0^{(-1)}(t) + \int_{\phi_0(\partial\mathbb{B}_n)} \frac{\partial}{\nu_{\phi_0(s)}}\left[S_n(t-s)\right]\tau\circ\phi_0^{(-1)}(s)\,d\sigma_s = 1$$

for all $t \in \phi_0(\partial\mathbb{B}_n)$, would have at least a solution $\tau \in L^2(\phi_0(\partial\mathbb{B}_n))$, a fact which is known to be false (cf. *e.g.*, Folland [9, Cor. 3.40, p. 184].) Hence,

$$\int_{\partial\mathbb{B}_n} \overline{\vartheta}_0\tilde{\sigma}[\phi_0]\,d\sigma \neq 0\,. \tag{26}$$

By classical Potential Theory, the first equation of (24) admits a solution $\overline{\vartheta}_1 \in C^{m-1,\alpha}(\partial\mathbb{B}_n)$ (cf. *e.g.*, Folland [9, Thm. 3.41, p. 185], Theorem 14 (ii) of Appendix C.) Then condition (26) ensures that there exists $k \in \mathbb{R}$ such that $\overline{\vartheta} \equiv \overline{\vartheta}_1 + k\overline{\vartheta}_0$ satisfies (24). The uniqueness for system (24) follows by the definition of $\overline{\vartheta}_0$ and by (26). □

Condition (21) shows that there is no hope to apply the Implicit Function Theorem to equation (19) around a zero of Ψ. Indeed, the linearized operator $\partial_\vartheta \Psi[\phi_0, g_0, \vartheta_0]$ has image $V^{m,\alpha}_{\phi_0}$, which is a proper subspace of the target space of Ψ, and which depends on ϕ_0. To circumvent this difficulty which did not appear for the Dirichlet problem, we resort to Theorem 13 of the Appendix B, which can be deduced by the Implicit Function Theorem, and we prove the following.

Theorem 9 *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. The map $\hat{\vartheta}[\cdot,\cdot]$ of $(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$ which takes (ϕ, g) to $\hat{\vartheta}[\phi, g]$ is real analytic.*

Proof We plan to apply Theorem 13 of Appendix B by taking F equal to Ψ and G equal to the function of $H \equiv (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)^3 \times \mathbb{R}$ to $\mathbb{R}$ defined by

$$G[\phi, g, \vartheta, \gamma, \omega] \equiv \int_{\partial\mathbb{B}_n} \gamma\tilde{\sigma}[\phi]\, d\sigma \qquad \forall (\phi, g, \vartheta, \gamma, \omega) \in H\,.$$

Obviously, G is real analytic and $G[\phi, g, \vartheta, 0, 0] = 0$ and $G[\phi, g, \vartheta, \Psi[\phi, g, \vartheta]] = 0$ for all $(\phi, g, \vartheta) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)^2$ (see (21)). Moreover, the partial differential $\partial_{(\gamma,\omega)} G[\phi, g, \hat{\vartheta}[\phi, g], 0, 0]$ coincides with the map which takes $(\overline{\gamma}, \overline{\omega})$ to $\int_{\partial\mathbb{B}_n} \overline{\gamma}\tilde{\sigma}[\phi]\, d\sigma$, which is surjective onto $\mathbb{R}$ and has kernel $V^{m,\alpha}_{\phi}$, which is a closed subspace of $C^{m-1,\alpha}(\partial\mathbb{B}_n) \times \mathbb{R}$ of codimension 1. Then the Theorem follows by Theorem 8, and by Theorem 13 of the Appendix B. □

Then we can write formula (16) as

$$\begin{aligned} v[\phi, g](t) \;=\; & \int_{\partial\mathbb{B}_n} \hat{\vartheta}[\phi, g](y) S_n(t - \phi(y))\tilde{\sigma}[\phi](y)\, d\sigma_y \\ & - \Big\{ \int_{\partial\mathbb{B}_n} \tilde{\sigma}[\phi]\, d\sigma \Big\}^{-1} \int_{\partial\mathbb{B}_n} \Big\{ \int_{\partial\mathbb{B}_n} \hat{\vartheta}[\phi, g](y) S_n(\phi(x) \\ & - \phi(y))\tilde{\sigma}[\phi](y)\, d\sigma_y \Big\} \tilde{\sigma}[\phi](x)\, d\sigma_x\,, \end{aligned} \tag{27}$$

for all $t \in \mathbb{I}[\phi]$. Then by Proposition 1, Theorems 7, 9, we deduce the validity of the following.

Theorem 10 *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let $(\phi_0, g_0) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$. Let Ω be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega \subseteq \mathbb{I}[\phi_0]$. Then there exists an open neighborhood $\mathcal{V}$ of (ϕ_0, g_0) in $(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$ such that*

(i) $\mathrm{cl}\Omega \subseteq \mathbb{I}[\phi]$ *for all* $(\phi, g) \in \mathcal{V}$.

(ii) The operator B_Ω of $\mathcal{V}$ to $C^0(\mathrm{cl}\Omega)$ which takes (ϕ, g) to the restriction $v[\phi, g]_{|\mathrm{cl}\Omega}$ of $v[\phi, g]$ to $\mathrm{cl}\Omega$ is real analytic.

Then by Proposition 2 and Theorem 10, we immediately deduce the validity of the following.

Theorem 11 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. The map B of*

$$\mathcal{G}^{m,\alpha} \equiv \{(\phi, g, t) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n) \times \mathbb{R}^n : t \in \mathbb{I}[\phi]\}$$

to $\mathbb{R}$ defined by

$$B[\phi, g, t] \equiv v[\phi, g](t) \qquad \forall (\phi, g, t) \in \mathcal{G}^{m,\alpha}$$

is real analytic.

We note that the subset of the triples (ϕ, g, t) of $\mathcal{G}^{m,\alpha}$ such that the compatibility condition $\int_{\phi(\partial\mathbb{B}_n)} g \circ \phi^{(-1)}\, d\sigma = 0$ holds can be shown to be a manifold of codimension one in $\mathcal{G}^{m,\alpha}$. Thus what Theorem 11 says is that on such a manifold the map B is real analytic. A similar Remark holds for Theorem 10. We now turn to consider the energy integral of the solution $v[\phi, g]$. As for the Dirichlet problem, we can prove the following.

Theorem 12 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. The functional $\mathcal{E}_N$ of $(C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$ to $\mathbb{R}$ defined by*

$$\mathcal{E}_N[\phi, g] \equiv \int_{\mathbb{I}[\phi]} |\nabla v[\phi, g](t)|^2\, dt\,. \tag{28}$$

for all $(\phi, g) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$ is real analytic.

Proof By the Divergence Theorem and by formula (27), we have

$$\begin{aligned}&\mathcal{E}_N[\phi, g]\\&= \int_{\partial\mathbb{B}_n} V[\phi, \hat{\vartheta}[\phi, g]] \left\{ g - \left\{ \int_{\partial\mathbb{B}_n} \tilde{\sigma}[\phi]\, d\sigma \right\}^{-1} \int_{\partial\mathbb{B}_n} g\tilde{\sigma}[\phi]\, d\sigma \right\} \tilde{\sigma}[\phi]\, d\sigma\,,\end{aligned}$$

for all $(\phi, g) \in (C^{m,\alpha}(\partial\mathbb{B}_n, \mathbb{R}^n) \cap \mathcal{A}_{\partial\mathbb{B}_n}) \times C^{m-1,\alpha}(\partial\mathbb{B}_n)$. Thus we can conclude by Proposition 1 and by Theorems 7, 9. □

Remark 1 Both in case of the Dirichlet problem and of the Neumann problem, one could exploit the representation formulas for the solutions and for the corresponding energy integrals, and the formulas for the differentials of simple and double layer potentials with respect to ϕ and the densities computed in Lanza and Rossi [22] to obtain formulas for all order differentials of solutions and energy integrals upon variation of ϕ and of the data.

3 Appendix A

In this Appendix we prove Proposition 2. To do so, we first introduce the Romieu classes. For all bounded open subsets Ω of $\mathbb{R}^n$ and $\rho > 0$, we set

$$C^0_{\omega,\rho}(\mathrm{cl}\Omega) \equiv \left\{ u \in C^\infty(\mathrm{cl}\Omega) : \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\Omega)} < +\infty \right\},$$

and

$$\|u\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)} \equiv \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\Omega)} \quad \forall u \in C^0_{\omega,\rho}(\mathrm{cl}\Omega)\,.$$

As is well known, the Romieu class $\left(C^0_{\omega,\rho}(\mathrm{cl}\Omega), \|\cdot\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)}\right)$ is a Banach space.

By standard interior estimates for harmonic functions (cf. *e.g.*, Gilbarg and Trudinger [10, Thm. 2.10, p. 23]), and by Stirling's inequality, the following Proposition holds.

Proposition 8 *Let Ω, Ω_1 be bounded open subsets of $\mathbb{R}^n$, $\mathrm{cl}\Omega_1 \subseteq \Omega$. Then there exists $\rho > 0$ such that $u_{|\mathrm{cl}\Omega_1} \in C^0_{\omega,\rho}(\mathrm{cl}\Omega_1)$ for all $u \in C^0_h(\mathrm{cl}\Omega)$ (cf. (9).) Moreover, the restriction operator is linear and continuous from $C^0_h(\mathrm{cl}\Omega)$ endowed with the norm of the uniform convergence in $\mathrm{cl}\Omega$ to $C^0_{\omega,\rho}(\mathrm{cl}\Omega_1)$.*

Then we can prove the following Proposition, which is just a slight variant of a special case of a corresponding result of Preciso [32, Prop. 1.1, p. 101].

Proposition 9 *Let Ω_1, Ω_2 be bounded open subsets of $\mathbb{R}^n$, Ω_2 connected. Let $\rho > 0$. Then the composition operator T of $C^0_{\omega,\rho}(\mathrm{cl}\Omega_1) \times C^0(\mathrm{cl}\Omega_2, \Omega_1)$ to $C^0(\mathrm{cl}\Omega_2)$ defined by*

$$T[u, v] \equiv u \circ v \quad \forall (u, v) \in C^0_{\omega,\rho}(\mathrm{cl}\Omega_1) \times C^0(\mathrm{cl}\Omega_2, \Omega_1)\,,$$

is real analytic.

Proof It suffices to show that for all (u_0, v_0) in the domain of T, there exists an open neighborhood $\mathcal{W}_0$ of (u_0, v_0) such that T is real analytic on $\mathcal{W}_0$. Let V be an open bounded connected neighborhood of class C^∞ of the compact subset $v_0(\mathrm{cl}\Omega_2)$ such that $\mathrm{cl}V \subseteq \Omega_1$ (cf. *e.g.*, [18, p. 935].) We now set $\mathcal{W}_0 \equiv C^0_{\omega,\rho}(\mathrm{cl}\Omega_1) \times C^0(\mathrm{cl}\Omega_2, V)$, and we show that T is real analytic on $\mathcal{W}_0$. We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of polynomial functions in n real variables with real coefficients. Then for each $r \in \mathbb{N}$, we take the closure $C^r_p(\mathrm{cl}V)$ of $\mathcal{P}(\mathbb{R}^n)$ in $C^r(\mathrm{cl}V)$. Since V is of class C^∞, we have $C^r_p(\mathrm{cl}V) = C^r(\mathrm{cl}V)$, and thus

$$\bigcap_{r \in \mathbb{N}} C^r_p(\mathrm{cl}V) = C^\infty(\mathrm{cl}V)\,.$$

Accordingly, we are in the conditions to invoke Proposition 2.16, p. 164 of [17] and conclude that T is real analytic from $C^0_{\omega,\rho}(\mathrm{cl}V) \times C^0(\mathrm{cl}\Omega_2, V)$ to $C^0(\mathrm{cl}\Omega_2)$. Since the restriction operator is linear and continuous from $C^0_{\omega,\rho}(\mathrm{cl}\Omega_1)$ to $C^0_{\omega,\rho}(\mathrm{cl}V)$, we conclude that T is real analytic on $\mathcal{W}_0$. □

Corollary 1 *Let Ω_1 be a bounded open subset of $\mathbb{R}^n$. Let $\rho > 0$. Then the map Ξ of $C^0_{\omega,\rho}(\mathrm{cl}\Omega_1) \times \Omega_1$ to $\mathbb{R}$ which takes (u, t) to $u(t)$ is real analytic.*

Proof By arguing in each connected component, we can assume that Ω_1 is connected. We first observe that the map of Ω_1 to $C^0(\mathrm{cl}\Omega_1, \Omega_1)$ which takes $t \in \Omega_1$ to the function b_t of $\mathrm{cl}\Omega_1$ to Ω_1 identically equal to t is real analytic. Then we observe that $\Xi[u, t] = T[u, b_t]$, and we apply Proposition 9. □

Proof of Proposition 2 of section 1 The statement follows immediately by Proposition 8 and by Corollary 1. □

4 Appendix B

In this appendix, we prove the following perhaps known consequence of the Implicit Function Theorem.

Theorem 13 *Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, $\mathcal{Z}_1$ be Banach spaces. Let $\mathcal{O}$ be an open subset of $\mathcal{X} \times \mathcal{Y}$, $(x_0, y_0) \in \mathcal{O}$, F a real analytic map of $\mathcal{O}$ to $\mathcal{Z}$, $F(x_0, y_0) = 0$. Let the partial differential $\partial_y F(x_0, y_0)$ with respect to the variable y be a linear homeomorphism of $\mathcal{Y}$ onto its image $V \equiv \operatorname{Im} \partial_y F(x_0, y_0)$. Assume that there exists a closed subspace V_1 of $\mathcal{Z}$ such that $\mathcal{Z} = V \oplus V_1$ algebraically. Let $\mathcal{O}_1$ be an open subset of $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ containing $(x_0, y_0, 0)$ such that $\mathcal{O}_1 \supseteq \{(x, y, F(x, y)) : (x, y) \in \mathcal{O}\}$, $\mathcal{O}_1 \supseteq \{(x, y, 0) : (x, y) \in \mathcal{O}\}$. Let G be a real analytic map of $\mathcal{O}_1$ to $\mathcal{Z}_1$ such that $G(x, y, F(x, y)) = 0$ for all $(x, y) \in \mathcal{O}$, $G(x, y, 0) = 0$ for all $(x, y) \in \mathcal{O}$, and such that the partial differential $\partial_z G(x_0, y_0, 0)$ is surjective onto $\mathcal{Z}_1$ and has kernel equal to V. Then there exist an open neighborhood $\mathcal{U}$ of x_0 in $\mathcal{X}$ and an open neighborhood $\mathcal{V}$ of y_0 in $\mathcal{Y}$ with $\mathcal{U} \times \mathcal{V} \subseteq \mathcal{O}$ and such that the set of zeros of F in $\mathcal{U} \times \mathcal{V}$ coincides with the graph of a real analytic function γ of $\mathcal{U}$ to $\mathcal{V}$.*

Proof Since V is homeomorphic to a Banach space, it is a closed subspace of $\mathcal{Z}$. Let π be the projection of $\mathcal{Z}$ onto V associated to the sum $\mathcal{Z} = V \oplus V_1$. Since both V and V_1 are closed, then π is continuous and thus the sum is also topological (cf. *e.g.*, Brezis [4, Teor. II.8].) Let j denote the natural injection of V_1 into $\mathcal{Z}$. By applying the Implicit Function Theorem for real analytic functions (cf. *e.g.*, Prodi and Ambrosetti [34]) to the function G_1 of $\mathcal{O}_1 \cap (\mathcal{X} \times \mathcal{Y} \times V_1)$ to $\mathcal{Z}_1$ defined by $G_1(x, y, z) = G(x, y, j(z))$ for all $(x, y, z) \in \mathcal{O}_1 \cap (\mathcal{X} \times \mathcal{Y} \times V_1)$ around the point $(x_0, y_0, 0)$, we deduce that there exist an open neighborhood $\mathcal{W}_1$ of (x_0, y_0) in $\mathcal{X} \times \mathcal{Y}$ and an open neighborhood $\mathcal{V}_1$ of 0 in V_1 with $\mathcal{W}_1 \times \mathcal{V}_1 \subseteq \mathcal{O}_1$, and a real analytic map γ_1 of $\mathcal{W}_1$ to $\mathcal{V}_1$ such that the graph of γ_1 coincides with the set of zeros of G_1 in $\mathcal{W}_1 \times \mathcal{V}_1$. Since $G(x, y, 0) = 0$ for all $(x, y) \in \mathcal{W}_1$, we conclude that γ_1 is identically

zero. Then we note that

$$\begin{aligned}\Big\{(x,y,z)\in\mathcal{O}_1:\,(x,y)\in\mathcal{W}_1, z-\pi z\in\mathcal{V}_1, G(x,y,z)=0, \pi(z)=0\Big\}\\ =\Big\{(x,y,z)\in\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}:\,(x,y)\in\mathcal{W}_1, (I-\pi)(z)=\gamma_1(x,y),\\ \pi(z)=0\Big\}\\ =\Big\{(x,y,z)\in\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}:\,(x,y)\in\mathcal{W}_1, z=0\Big\}. \qquad (29)\end{aligned}$$

Next we apply the Implicit Function Theorem to the function $\pi\circ F$ of $\mathcal{O}$ to V around (x_0,y_0), and we conclude that there exist an open neighborhood $\mathcal{U}$ of x_0 in $\mathcal{X}$ and an open neighborhood $\mathcal{V}$ of y_0 in $\mathcal{Y}$ with $\mathcal{U}\times\mathcal{V}\subseteq\mathcal{W}_1\cap\mathcal{O}$, $(I-\pi)\circ F(\mathcal{U}\times\mathcal{V})\subseteq\mathcal{V}_1$ and a real analytic map γ of $\mathcal{U}$ to $\mathcal{V}$ such that the graph of γ coincides with the set of zeros of $\pi\circ F$ in $\mathcal{U}\times\mathcal{V}$. Now we note that if $(x,y)\in\mathcal{U}\times\mathcal{V}$ and $\pi\circ F(x,y)=0$, then condition $G(x,y,F(x,y))=0$ and (29) imply that $F(x,y)=0$. Then the set of zeros of F in $\mathcal{U}\times\mathcal{V}$ coincides with the set of zeros of $\pi\circ F$ in $\mathcal{U}\times\mathcal{V}$, which in turn coincides with the graph of γ. □

Clearly, a corresponding statement holds if F, G are of class C^k for some $0<k\in\mathbb{N}\cup\{\infty\}$, a case in which the corresponding function γ would be of class C^k.

5 Appendix C

In this Appendix we present a probably known slight variant of a classical result in Potential Theory, which we prove by a standard argument, and for which the author takes no credit.

Theorem 14 *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let Ω be a bounded connected open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let ν denote the unit outward normal to Ω on $\partial\Omega$. Then the following statements hold.*

(i) If $\Gamma\in C^{m,\alpha}(\partial\Omega)$, $\mu\in C^0(\partial\Omega)$ and

$$\Gamma(t)=\frac{1}{2}\mu(t)+\int_{\partial\Omega}\mu(s)\frac{\partial}{\partial\nu(s)}\left[S_n(t-s)\right]d\sigma_s \qquad \forall t\in\partial\Omega\,, \quad (30)$$

then $\mu\in C^{m,\alpha}(\partial\Omega)$.

(ii) If $\Gamma \in C^{m-1,\alpha}(\partial\Omega)$, $\mu \in C^0(\partial\Omega)$ *and*

$$\Gamma(t) = -\frac{1}{2}\mu(t) + \int_{\partial\Omega} \mu(s) \frac{\partial}{\partial\nu(t)} \left[S_n(t-s)\right] d\sigma_s \qquad \forall t \in \partial\Omega\,, \tag{31}$$

then $\mu \in C^{m-1,\alpha}(\partial\Omega)$.

Proof We first prove statement (i). By Miranda [25, §15, II, VI] and by equation (30) and the membership of Γ in $C^{1,\alpha}(\partial\Omega)$, we have $\mu \in C^{1,\alpha}(\partial\Omega)$. Now we set

$$w(t) \equiv \int_{\partial\Omega} \mu(s) \frac{\partial}{\partial\nu(s)} \left[S_n(t-s)\right] d\sigma_s \qquad \forall t \in \mathbb{R}^n \setminus \partial\Omega\,,$$

and we denote by w^+ the continuous extension to $\mathrm{cl}\Omega$ of $w_{|\Omega}$ and by w^- the continuous extension to $\mathbb{R}^n \setminus \Omega$ of $w_{|\mathbb{R}^n \setminus \mathrm{cl}\Omega}$. Let $R > 0$ be such that $\mathrm{cl}\Omega \subseteq R\mathbb{B}_n$, and $\Omega_R \equiv (R\mathbb{B}_n) \setminus \mathrm{cl}\Omega$. By classical Potential Theory we have $w^+ \in C^{1,\alpha}(\mathrm{cl}\Omega)$ and $w^- \in C^{1,\alpha}(\mathrm{cl}\Omega_R)$ (cf. *e.g.*, Lanza and Rossi [22, Thm. 3.1].) Since $w^+_{|\partial\Omega} = \Gamma \in C^{m,\alpha}(\partial\Omega)$, classical elliptic regularity theory for the Dirichlet problem implies that $w^+ \in C^{m,\alpha}(\mathrm{cl}\Omega)$ (cf.*e.g.*, Gilbarg and Trudinger [10, Thms. 6.19, 8.34].) By standard jump conditions for double layer potentials, we have $\frac{\partial w^-}{\partial\nu} = \frac{\partial w^+}{\partial\nu}$ on $\partial\Omega$. Then we have $\frac{\partial w^-}{\partial\nu} \in C^{m-1,\alpha}(\partial\Omega)$. Since we also have $\frac{\partial w^-}{\partial\nu} \in C^\infty(\partial(R\mathbb{B}_n))$, classical elliptic regularity theory for the Neumann problem implies that $w^-_{|\mathrm{cl}\Omega_R} \in C^{m,\alpha}(\mathrm{cl}\Omega_R)$ (cf. *e.g.*, Miranda [25, §16, II], Troianiello [36, Thms. 1.17 (ii), 3.16 (iii)], Agmon, Douglis and Nirenberg [1, Thm. 7.3].) Then by the jump relation $w^+ - w^- = \mu$ on $\partial\Omega$, we conclude that $\mu \in C^{m,\alpha}(\partial\Omega)$.

We now prove statement (ii) by a similar standard argument. By Miranda [25, §14, IV], we have $\mu \in C^{0,\beta}(\partial\Omega)$ for $0 < \beta < \alpha$. Now we set

$$v(t) \equiv \int_{\partial\Omega} \mu(s) S_n(t-s)\, d\sigma_s \qquad \forall t \in \mathbb{R}^n\,,$$

and we denote by v^+ the continuous extension to $\mathrm{cl}\Omega$ of $v_{|\Omega}$ and by v^- the continuous extension to $\mathbb{R}^n \setminus \Omega$ of $v_{|\mathbb{R}^n \setminus \mathrm{cl}\Omega}$. By classical Potential Theory, we have $v^+ \in C^{1,\beta}(\mathrm{cl}\Omega)$, $v^- \in C^{1,\beta}(\mathrm{cl}\Omega_R)$ (cf. *e.g.*, Lanza and Rossi [22, Thm. 3.1].) Since $\frac{\partial v^+}{\partial\nu} = g \in C^{m-1,\alpha}(\partial\Omega)$, the above mentioned regularity results for the Neumann problem imply that $v^+ \in C^{m,\alpha}(\mathrm{cl}\Omega)$. Since $v^- = v^+$ on $\partial\Omega$ and $v_{|\partial(R\mathbb{B}_n)} \in C^\infty(\partial(R\mathbb{B}_n))$, the above mentioned regularity results for the Dirichlet problem imply

that $v^- \in C^{m,\alpha}(\mathrm{cl}\Omega)$. Then by the jump relation $\frac{\partial v^-}{\partial \nu} - \frac{\partial v^+}{\partial \nu} = \mu$ on $\partial\Omega$, we have $\mu \in C^{m-1,\alpha}(\partial\Omega)$. □

We note that Theorem 14 still holds if the sign in front of the factor $\frac{1}{2}$ in equations (30), (31) is changed. Also, we note that the conclusions of statements (i), (ii) of Theorem 14 with $\mu \in C^{m,\alpha}(\partial\Omega)$ and $\mu \in C^{m-1,\alpha}(\partial\Omega)$ replaced by $\mu \in C^{m,\alpha_1}(\partial\Omega)$ and $\mu \in C^{m-1,\alpha_1}(\partial\Omega)$ for some $0 < \alpha_1 < \alpha$, respectively and with $n = 3$ can be found in the classical book of Günter [11].

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The optimization of fixed point methods

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Abstract

Subject of the consideration are fixed point equations of the form

$$u = u_0 + \mathcal{A}\mathcal{F}u$$

in a Banach space $\mathcal{B}$. Two operators $\mathcal{A}$ and $\mathcal{F}$ are defined in the entire space $\mathcal{B}$ mapping $\mathcal{B}$ into itself. While $\mathcal{A}$ is linear and bounded $\mathcal{F}$ in general is assumed to be non-linear but Lipschitz continuous on bounded subsets of $\mathcal{B}$. The Lipschitz constant, depending on the bounded subset of $\mathcal{B}$ are not necessarily bounded when the bounded subset is enlarging. The element $u_0 \in \mathcal{B}$ is a fixed element. In applications to initial or boundary value problems for ordinary and partial differential equations, see [1]-[4], $\mathcal{A}$ is an integral operator and $\mathcal{F}$ is given by the right-hand side of a differential equation $\mathcal{L}u = \mathcal{F}u$. Here $\mathcal{L}$ is a linear differential operator which can be inverted by an integral operator. The fixed point equation involved is solved by the contracting mapping principle. As $\mathcal{F}$ in general fails to be Lipschitz continuous on the entire space B the fixed point principle can only be applied on suitable bounded subsets rather than in $\mathcal{B}$ itself. Proper subsets are balls or polydiscs. The Lipschitz constant will grow when the subset is enlarged. Thus the question arrises how large these subsets have to be chosen such that the norm $\|\mathcal{A}\|$ of $\mathcal{A}$ only leads to minor restrictions. For a two-dimensional problem we will prove that in the case of two equal Lipschitz constants there exists a uniquely determined optimal polycylinder.

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Keywords: Two component fixed point equations in Banach spaces, contraction principle, optimal polycylinder

1 Introduction and result

Fixed point equations of the form

$$u = u_0 + \mathcal{A}\mathcal{F}u$$

are considered in a Banach space $\mathcal{B}$. Here $\mathcal{A}$ and $\mathcal{F}$ are operators mapping $\mathcal{B}$ into itself. In general $\mathcal{F}$ is non-linear but Lipschitz continuous on bounded subsets of $\mathcal{B}$. In applications to initial value problems for ordinary or partial differential equations, see [1]-[4], $\mathcal{A}$ is an integral operator and $\mathcal{F}$ is given by the right-hand side of a differential equation $\mathcal{L}u = \mathcal{F}u$. Here $\mathcal{L}$ is a linear differential operator, which can be inverted by an integral operator. The fixed point equation can be solved by the contraction principle. As in general $\mathcal{F}$ fails to be Lipschitz continuous in the entire space $\mathcal{B}$ the fixed point principle can be applied only in bounded subsets. Of course the Lipschitz constants grow when the subsets become larger. Hence the question arises, how large these subsets can be chosen so that the norm $\|\mathcal{A}\|$ of $\mathcal{A}$ may become as large as possible. In the applications to initial value problems for ordinary differential equations the magnitude of $\|\mathcal{A}\|$ has some influence on the extension of the interval where the solution exists. Let $\mathcal{B}$ be an arbitrary Banach space with norm $\|\cdot\|$. By $u_1, u_2, u_3, \ldots$ we denote elements of $\mathcal{B}$. Let $\mathcal{X} = \mathcal{B} \times \mathcal{B}$ the Banach space of pairs (u_1, u_2) with

$$\|(u_1, u_2)\| = \max(\|u_1\|, \|u_2\|) \ .$$

Furthermore let $(F_1(\cdot, u_1, u_2), F_2(\cdot, u_1, u_2))$ a mapping from $\mathcal{X} = \mathcal{B} \times \mathcal{B}$ into itself, where F_1 and F_2 fulfill a Lipschitz condition of the form

$$\|F_j(\cdot, \tilde{u_1}, \tilde{u_2}) - F_j(\cdot, \hat{u_1}, \hat{u_2})\| \leq L_{j1}\|\tilde{u_1} - \hat{u_1}\| + L_{j2}\|\tilde{u_2} - \hat{u_2}\|, \ \ j = 1, 2.$$

Finally let $\mathcal{A}_1$ and $\mathcal{A}_2$ be linear and bounded operators, mapping $\mathcal{B}$ into itself. With these quantities a system of operator equations of the form

$$\begin{aligned} u_1 &= u_{01} + \mathcal{A}_1 F_1(\cdot, u_1, u_2), \\ u_2 &= u_{02} + \mathcal{A}_2 F_2(\cdot, u_1, u_2) \end{aligned}$$

is considered. The contraction principle is applied with respect to the polydisc

$$\mathcal{P} = \{(u_1, u_2) \in \mathcal{B} \times \mathcal{B} : \|u_j - u_{0j}\| \leq d_j, j = 1, 2\}$$

with the center (u_{01}, u_{02}). Denoting $\|F_j(\cdot, u_{01}, u_{02})\| = M_j$ then

$$\begin{aligned}\|\mathcal{A}_1\|(L_{11}d_1 + L_{12}d_2 + M_1) &\leq d_1,\\ \|\mathcal{A}_2\|(L_{21}d_1 + L_{22}d_2 + M_2) &\leq d_2,\end{aligned}$$

$$\begin{aligned}\|\mathcal{A}_1\|(L_{11} + L_{12}) &< 1,\\ \|\mathcal{A}_2\|(L_{21} + L_{22}) &< 1,\end{aligned}$$

follow. The following problem is studied: The bounds for the norms of the operators should be maximal, i.e

$$\min(\|A_1\|, \|A_2\|)$$

should become as large as possible. This will be achieved by the radii d_1,d_2 of the polydisc. There is the optimization problem: Given functions $L_{jk}(d_1, d_2)$ for $d_1, d_2 > 0$ such that $\min(\|A_1\|, \|A_2\|)$ becomes maximal for the solution of the inequality system

$$\begin{aligned}\|\mathcal{A}_1\| &\leq \frac{d_1}{L_{11}d_1 + L_{12}d_2 + M_1},\\ \|\mathcal{A}_1\| &< \frac{1}{L_{11} + L_{12}},\\ \|\mathcal{A}_2\| &\leq \frac{d_2}{L_{21}d_1 + L_{22}d_2 + M_2},\\ \|\mathcal{A}_2\| &< \frac{1}{L_{21} + L_{22}}.\end{aligned} \qquad (*)$$

For related considerations see [1]-[4].
It has to be observed that the above Lipschitz constants are growing when the polydisc becomes larger. Therefore the Lipschitz constants are assumed to be positive and monotonically increasing. In the sequel $L_{11} = L_{21} = L_1$,$L_{12} = L_{22} = L_2$ are assumed and the right-hand sides of $(*)$ are denoted by h_1, h_2, h_3. Moreover, all first order partial derivatives of L_1 and L_2 are assumed to be positive and the mixed partial derivatives are assumed to be non negative. Then with

$$\begin{aligned}h_1(d_1, d_2) &= \frac{d_1}{L_1d_1 + L_2d_2 + M_1},\\ h_2(d_1, d_2) &= \frac{d_2}{L_1d_1 + L_2d_2 + M_2},\\ h_3(d_1, d_2) &= \frac{1}{L_1 + L_2},\end{aligned} \qquad (1)$$

$h_3\,(d_1, d_2)$ can not be the minimum among them. Since the two systems of equations

$$\begin{aligned} \frac{\partial h_1}{\partial d_1} = 0 \quad &\Rightarrow \qquad M_1 + L_2 d_2 - \frac{\partial L_1}{\partial d_1}\, d_1^2 - \frac{\partial L_2}{\partial d_1}\, d_1 d_2 = 0 \\ \frac{\partial h_1}{\partial d_2} = 0 \quad &\Rightarrow \qquad -d_1 \left(\frac{\partial L_1}{\partial d_2}\, d_1 + \frac{\partial L_2}{\partial d_2}\, d_2 + L_2 \right) = 0 \end{aligned} \tag{I}$$

and

$$\begin{aligned} \frac{\partial h_2}{\partial d_1} = 0 \quad &\Rightarrow \qquad -d_2 \left(\frac{\partial L_1}{\partial d_1}\, d_1 + \frac{\partial L_2}{\partial d_1}\, d_2 + L_1 \right) = 0 \\ \frac{\partial h_2}{\partial d_2} = 0 \quad &\Rightarrow \qquad M_2 + L_1 d_1 - \frac{\partial L_1}{\partial d_2}\, d_1 d_2 - \frac{\partial L_2}{\partial d_2}\, d_2^2 = 0 \end{aligned} \tag{II}$$

have no solutions for $d_1, d_2 > 0$, we look for the maximum of $h_1(d_1, d_2)$ on the curve γ,

$$\gamma = \{(d_1, d_2) \in \mathbb{R}_+^2 \mid h_1(d_1, d_2) = h_2(d_1, d_2)\} \tag{2}$$

*E*xample for γ: Assuming $M_1 < M_2$, we set $L_1 = \kappa_1$, $L_2 = \kappa_2$ with $\kappa_1, \kappa_2 \in \mathbb{R}$, $\kappa_1, \kappa_2 > 0$. Then γ takes the form

$$(d_1 - d_2)(\kappa_1 d_1 + \kappa_2 d_2 + M_2) + d_2(M_2 - M_1) = 0$$

and can be given in explicit form:

$$\gamma: \quad d_2 = \frac{(\kappa_2 - \kappa_1)d_1 - M_1 + \sqrt{((\kappa_2 - \kappa_1)d_1 - M_1)^2 + 4\kappa_2(\kappa_1 d_1^2 + M_2 d_1)}}{2\kappa_2}.$$

Then we obtain

$$h_{1\gamma} := h_1|_\gamma = \frac{-(\kappa_1 d_1 + \kappa_2 d_2 + M_1) + \sqrt{((\kappa_2 - \kappa_1)d_1 - M_1)^2 + 4\kappa_2(\kappa_1 d_1^2 + M_2 d_1)}}{2\kappa_2(M_2 - M_1)},$$

where $h_{1\gamma}(d_1 = 0) = 0$, $\lim\limits_{d_1 \to +\infty} h_{1\gamma} = \frac{1}{\kappa_1 + \kappa_2}$ and $h'_{1\gamma}(d_1) > 0$ for all $d_1 > 0$.

This means that d_1 can be chosen arbitrarily large in this case. Making different assumptions, we shall prove now the following proposition:

Theorem *If the partial derivatives of L_1 and L_2 satisfy the inequalities*

$$\frac{\partial L_1}{\partial d_1} > 0, \quad \frac{\partial L_1}{\partial d_2} > 0, \quad \frac{\partial L_2}{\partial d_1} > 0, \quad \frac{\partial L_2}{\partial d_2} > 0,$$
$$\frac{\partial^2 L_1}{\partial {d_1}^2} \geq 0, \quad \frac{\partial^2 L_1}{\partial d_1 \partial d_1} \geq 0, \quad \frac{\partial^2 L_1}{\partial d_2 \partial d_1} \geq 0, \quad \frac{\partial^2 L_1}{\partial {d_2}^2} \geq 0,$$
$$\frac{\partial^2 L_2}{\partial {d_1}^2} \geq 0, \quad \frac{\partial^2 L_2}{\partial d_1 \partial d_1} \geq 0, \quad \frac{\partial^2 L_2}{\partial d_2 \partial d_1} \geq 0, \quad \frac{\partial^2 L_2}{\partial {d_2}^2} \geq 0,$$

then the objective function has one and only one global optimum and there exists a uniquely determined optimal polycylinder.

2 Proof of theorem

Case 1: $M_1 = M_2 = M > 0$. Then (2) is identical with

$$\frac{d_1}{L_1 d_1 + L_2 d_2 + M} = \frac{d_2}{L_1 d_1 + L_2 d_2 + M}$$

i.e.

$$0 = (d_1 - d_2)(L_1 d_1 + L_2 d_2 + M).$$

Because of $L_1 d_1 + L_2 d_2 + M > 0$, this implies $d_1 = d_2$. Let therefore $d := d_1 = d_2$; then we obtain

$$h_{1\gamma} = \frac{d}{(L_1 + L_2)d + M},$$
$$h'_{1\gamma} = \frac{M - d^2 \left(\frac{\partial L_1}{\partial d_1} + \frac{\partial L_1}{\partial d_2} + \frac{\partial L_2}{\partial d_1} + \frac{\partial L_2}{\partial d_2} \right)}{((L_1 + L_2)d + M)^2},$$

yielding the implication

$$h'_{1\gamma} = 0 \quad \Rightarrow \quad \frac{M}{d^2} = \frac{\partial L_1}{\partial d_1} + \frac{\partial L_1}{\partial d_2} + \frac{\partial L_2}{\partial d_1} + \frac{\partial L_2}{\partial d_2}.$$

If $L(d) := \frac{\partial L_1}{\partial d_1} + \frac{\partial L_1}{\partial d_2} + \frac{\partial L_2}{\partial d_1} + \frac{\partial L_2}{\partial d_2}$ has a positive second derivative, then there is exactly one maximum.

Case 2: $M_1 \neq M_2$. Without loss of generality let $M_1 < M_2$. Then (2) yields

$$\frac{d_1}{L_1 d_1 + L_2 d_2 + M_1} = \frac{d_2}{L_1 d_1 + L_2 d_2 + M_2}$$

i.e.

$$(d_1 - d_2)(L_1 d_1 + L_2 d_2 + M_2) + d_2(M_2 - M_1) = 0$$

which implies

$$d_1 - d_2 < 0 \quad \text{on } \gamma \setminus \{(0,0)\} \tag{3}$$

since $M_2 > M_1$.

To rewrite γ in another form, we set $g(d_1, d_2) := L_1 d_1 + L_2 d_2$. Then we have

$$\frac{d_1}{g + M_1} = \frac{d_2}{g + M_2} \quad \Leftrightarrow \quad g = \frac{M_1 d_2 - M_2 d_1}{d_1 - d_2}.$$

With $f(d_1, d_2) := \dfrac{M_1 d_2 - M_2 d_1}{d_1 - d_2}$, (2) is equivalent to

$$\gamma = \{(d_1, d_2) \in \mathbb{R}^2_+ \mid g(d_1, d_2) = f(d_1, d_2)\}.$$

Now we calculate $h_{1\gamma}$:

$$h_{1\gamma} = \frac{d_1}{g(d_1, d_2) + M_1} = \frac{d_1}{f(d_1, d_2) + M_1} = \frac{d_2 - d_1}{M_2 - M_1}. \tag{4}$$

As a result, $h_{1\gamma}$ takes its maximal value if $d_2 - d_1$ becomes maximal. If there is any such maximum, then γ is tangent to the straight line $d_2 - d_1 = c_M$.

Remark 1 Whenever a straight line $d_2 - d_1 = c$ with $c \in \mathbb{R}$, $0 < c < +\infty$, is tangent to γ, it yields a local maximum of h_1 in the point of contact.

Since we have $0 < h_1 < 1/L_1$, we can estimate the range where γ lies.

$$h_1|_\gamma = \frac{d_2 - d_1}{M_2 - M_1} < \frac{1}{L_1} \quad \Rightarrow \quad d_2 - d_1 < \frac{M_2 - M_1}{L_1} = k(d_1, d_2)$$

With (3) it follows that

$$d_2 - k(d_1, d_2) < d_1 < d_2. \tag{5}$$

Since unlimited growth of $k(d_1, d_2)$ is not possible, γ lies within a certain stripe.

The question arises whether there exists an extremal straight line

$$d_2 - d_1 = c_{\mathrm{M}}$$

which is tangent to γ, but does not intersect γ.

If not, then h_1 is bound to be the greater, the greater d_1 and h_2 are chosen on γ.

Further, if such a straight line exists - can there be more than one point of contact, i.e. more than one global maximum of h_1 ?

We turn to answer the first question. If $L_1(d_1, d_2) \to +\infty$ holds for $d_1 \to +\infty$, then one has $k(d_1, d_2) \to 0$, too. Because of $d_2 - k\,(d_1, d_2) < d_1 < d_2$ this means that γ approaches asymptotically to the straight line $d_1 = d_2$ as d_1 goes to $+\infty$. In this case, there must be at least one maximum $d_2 - d_1 = c_{\mathrm{M}}$ on γ.

As to the second question, consider

$$f(d_1, d_2) = \frac{d_2 M_1 - M_2 d_1}{d_1 - d_2}$$

within the range of angles $0 < d_1 < d_2$.

For $d_2 \to d_1 + 0$, one has $f(d_1, d_2) \to +\infty$ and $g(d_1, d_2) \to (L_1 + L_2)d_1 < +\infty$. Thus,

$$f(d_1, d_2) \begin{cases} < g(d_1, d_2) & \text{for } (d_1, d_2) \text{ above } \gamma, \\ = g(d_1, d_2) & \text{for } (d_1, d_2) \in \gamma, \\ > g(d_1, d_2) & \text{for } (d_1, d_2) \text{ below } \gamma. \end{cases}$$

In particular, $f \leq g$ for $d_2 - d_1 = c_{\mathrm{M}}$. (Equality holds in the point of contact only.)

Restricted to any straight line $d_2 - d_1 = c$, $c \in \mathbb{R}$, $0 < c < +\infty$, the function $f(d_1, d_2)$ is linear since

$$f(d_1, d_2) = f(d_1, d_1 + c) = \frac{(M_2 - M_1)d_1}{c} - M_1\,;$$

thus $f(d_1, d_2)$ is also linear along the extremal straight line $d_2 - d_1 = c_{\mathrm{M}}$. Now the function $g(d_1, d_2)$ is always convex when restricted to a straight line $d_2 - d_1 = c$ as above. This implies that the two functions f and g, restricted to the line $d_2 = d_1 + c_{\mathrm{M}}$, have one and only one point of contact. If f would have two intersection points with a convex g on such a straight line, then $f > g$ would hold in the segment between these two points which can't be the case if the straight line was chosen to be the extremal one, $d_2 = d_1 + c_{\mathrm{M}}$.

To determine the extreme value c_{M}, an extreme value problem with a constraint has to be solved, namely

$$h_1(d_1, d_2) = \frac{d_1}{L_1 d_1 + L_2 d_2 + M_1} \to \max \tag{6}$$

with the constraint

$$g(d_1, d_2) - f(d_1, d_2) = 0. \tag{7}$$

Instead of $h_1(d_1, d_2)$, we deal with the function

$$\tilde{h}_1(d_1, d_2) := \frac{d_2 - d_1}{M_2 - M_1}.$$

Since $h_1 = \tilde{h}_1$ on γ (see equation (4)), the maximal values of $\tilde{h}_1$ under the constraint $g\,(d_1, d_2) - f\,(d_1, d_2) = 0$ coincide with the solution of the original extreme value problem,

$$-\frac{1}{M_2 - M_1} + \lambda \left(\frac{\partial g}{\partial d_1} - \frac{\partial f}{\partial d_1} \right) = 0, \tag{8}$$

$$\frac{1}{M_2 - M_1} + \lambda \left(\frac{\partial g}{\partial d_2} - \frac{\partial f}{\partial d_2} \right) = 0, \tag{9}$$

$$g - f = 0, \tag{10}$$

$$\frac{\partial g}{\partial d_1} + \frac{\partial g}{\partial d_2} = \frac{\partial f}{\partial d_1} + \frac{\partial f}{\partial d_2}. \tag{11}$$

Remark 2 For the curve γ the equality

$$d_2' = \frac{d_2(M_2 - M_1) - (d_1 - d_2)^2 \left(\frac{\partial L_1}{\partial d_1} d_1 + L_1 + \frac{\partial L_2}{\partial d_1} d_2 \right)}{(d_1 - d_2)^2 \left(\frac{\partial L_1}{\partial d_2} d_1 + \frac{\partial L_2}{\partial d_2} d_2 + L_2 \right) + (M_2 - M_1) d_1}$$

holds. Here the denominator is always positive because of $M_2 > M_1$, $\dfrac{\partial L_1}{\partial d_2}, \dfrac{\partial L_2}{\partial d_2}$, $L_2 > 0$ and $d_1 > 0$, $d_2 > 0$. The curve is uniquely determined.

Remark 3 The function $g(d_1, d_2)$ restricted to any straight line $d_2 - d_1 = c$, $c > 0$ constant, fulfils

$$g(d_1, d_1 + c) = L_1(d_1, d_1 + c) d_1 + L_2(d_1, d_1 + c)(d_1 + c).$$

Since the right-hand side hereof depends on d_1 only, we can calculate its derivatives. The first derivative is

$$\frac{\partial L_1}{\partial d_1}(d_1, d_1 + c) d_1 + \frac{\partial L_1}{\partial d_2}(d_1, d_1 + c) d_1 + L_1(d_1, d_1 + c) + \\ \frac{\partial L_2}{\partial d_1}(d_1, d_1 + c)(d_1 + c) + \frac{\partial L_2}{\partial d_2}(d_1, d_1 + c)(d_1 + c) + L_2(d_1, d_1 + c)$$

The second derivative is

$$\left(\frac{\partial^2 L_1}{\partial {d_1}^2}(d_1, d_1+c) + \frac{\partial^2 L_1}{\partial d_1 \partial d_2}(d_1, d_1+c) + \frac{\partial^2 L_1}{\partial d_2 \partial d_1}(d_1, d_1+c) + \frac{\partial^2 L_1}{\partial {d_2}^2}(d_1, d_1+c)\right) d_1 + 2\left(\frac{\partial L_1}{\partial d_1}(d_1, d_1+c) + \frac{\partial L_1}{\partial d_2}(d_1, d_1+c)\right) + \left(\frac{\partial^2 L_2}{\partial {d_1}^2}(d_1, d_1+c) + \frac{\partial^2 L_2}{\partial d_1 \partial d_2}(d_1, d_1+c) + \frac{\partial^2 L_2}{\partial d_2 \partial d_1}(d_1, d_1+c) + \frac{\partial^2 L_2}{\partial {d_2}^2}(d_1, d_1+c)\right)(d_1+c) + 2\left(\frac{\partial L_2}{\partial d_1}(d_1, d_1+c) + \frac{\partial L_2}{\partial d_2}(d_1, d_1+c)\right);$$

it is positive if the inequalities

$$\begin{array}{llll} \frac{\partial L_1}{\partial d_1} > 0, & \frac{\partial L_1}{\partial d_2} > 0, & \frac{\partial L_2}{\partial d_1} > 0, & \frac{\partial L_2}{\partial d_2} > 0, \\ \frac{\partial^2 L_1}{\partial {d_1}^2} \geq 0, & \frac{\partial^2 L_1}{\partial d_1 \partial d_1} \geq 0, & \frac{\partial^2 L_1}{\partial d_2 \partial d_1} \geq 0, & \frac{\partial^2 L_1}{\partial {d_2}^2} \geq 0, \\ \frac{\partial^2 L_2}{\partial {d_1}^2} \geq 0, & \frac{\partial^2 L_2}{\partial d_1 \partial d_1} \geq 0, & \frac{\partial^2 L_2}{\partial d_2 \partial d_1} \geq 0, & \frac{\partial^2 L_2}{\partial {d_2}^2} \geq 0, \end{array}$$

hold. This completes the proof of the theorem.

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On decompositions of the complex quaternion valued Hilbert space related to the Helmholtz equation

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Abstract

The space $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ of complex quaternion-valued functions in a bounded domain Ω of $\mathbb{R}^3$ is decomposed as well into the orthogonal sum of the subspace of *poly-left $\alpha-$hyperholomorphic functions* of arbitrary order $k \geq 1$ and its orthogonal complement as into the orthogonal sum of the subspace of *poly-metaharmonic functions* of arbitrary order $k \geq 1$ and its orthogonal complement. The proofs are based on Stokes formulas and the existence of an inner product in $L_2(\Omega, \mathbb{H}(\mathbb{C}))$. In addition, the projections onto subspaces of metaharmonic functions are defined. The close connection of these projections with boundary value problems for bimetaharmonic functions are outlined.
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1 Introduction

One of the most interesting facts of complex and hypercomplex function theory is the orthogonal decomposition of the space $L_2(\Omega)$,

$$L_2(\Omega) = \ker D \cap L_2(\Omega) \oplus \overset{0}{D}(W_2^1(\Omega)) \tag{1}$$

where $\ker D(\Omega)$ denotes the set of all *holomorphic functions* The classical *monogenic functions* in Ω are also considered. This decomposition has a lot of applications, especially to the theory of partial differential equation, see for example [3, 5], and to the Stokes system as given, e.g., in [4, 6]. We also refer to [12] for extending the orthogonal decomposition (1) to the spaces $L_p(\Omega),\ 1 < p < \infty$.

We refer the readers to [2] for another proof of decompositions of Sobolev spaces in Clifford analysis. In this paper, the orthogonal complement of the subspace of *poly-left monogenic functions* of arbitrary order $k \geq 1$ and of the subspace of *polyharmonic functions* of arbitrary order $k \geq 1$ are detemined. The proofs are based on proper higher-order Cauchy-Pompeiu formulas and Green functions for powers of the Laplacian.

A decomposition of the space $L_2(\Omega)$ with respect to *left-$\alpha-$hyper-holomorphic functions* in real quaternionic analysis was given in [8, Chap.4], for Clifford analysis see [9, 14, 15]. The proofs of these decompositions are based on the existence of an inner product in $L_2(\Omega)$ and the properties of the boundary projections P_α, Q_α. Here, we investigate the orthogonal decompositions in complex quaternionic analysis. Besides decompositions with respect to *left $\alpha-$hyper-holomorphic functions,* decompositions are related to *poly-left $\alpha-$hyperholomorphic functions* as well as to *polymetaharmonic functions* are available. Our method to prove these decomposition is based on the Stokes formulas and the existence of an inner product in $L_2(\Omega, \mathbb{H}(\mathbb{C}))$.

The integral representations in terms of powers of the Helmholtz operator was proved in [10] and some important properties of the higher order Teodorescu operator also given, see [11]. Therefore, we can derive the projections onto the suspaces of metaharmonic functions. It opens the door for further consideration of boundary value problems of partial differential equations. In this paper, we also give one application to the Dirichlet problem of bimetaharmonic functions. The question of existence of the solution is answered in this section as well as an important convenient representation of the solutions. We

hope these integral representation formulas are adapted to the necessary numerical evaluation of the solutions which can be connected with mathematical models.

2 Preliminaries

Let $\{e_0, e_1, e_2, e_3\}$ be an orthonormal basis of $\mathbb{R}^4$ such that $x \in \mathbb{R}^4$ is represented as

$$x = \sum_{k=0}^{3} x_k e_k, \qquad x_k \in \mathbb{R}, \quad 0 \leq k \leq 3. \tag{2}$$

The part $x_0 e_0 =: Sc(x)$ is called the scalar part of x and $\vec{x} = \sum_{k=1}^{3} x_k e_k =: \mathrm{Vec}(x)$ the vector part of x.
A product is defined in $\mathbb{R}^4$ which satisfies the conditions

(i) $e_1^2 = e_2^2 = e_3^2 = -1$

(ii) $e_1 e_2 = -e_2 e_1 = e_3;\ e_2 e_3 = -e_3 e_2 = e_1;\ e_3 e_1 = -e_1 e_3 = e_2.$

The element e_0 is regarded as the usual unit, that is, $e_0 = 1$. Then, the algebraic rules (i), (ii) yield the quaternionic product

$$xy = x_0 y_0 - < \vec{x}, \vec{y} > + x_0 \vec{y} + \vec{x} y_0 + [\vec{x} \times \vec{y}]. \tag{3}$$

We are now prepared to give the definition of the algebra of real quaternions.

Definition 1 *The tuple $(\mathbb{R}^4, \cdot)$ is called the algebra of real quaternions. We signify $(\mathbb{R}^4, \cdot)$ by $\mathbb{H}(\mathbb{R})$.*

The quaternion $\bar{x} = x_0 - \vec{x}$ is called the conjugate to x. The number $|x|$ defined by

$$|x|^2 := x\bar{x} \tag{4}$$

is named the absolute value of x.

If now in the definition of a quaternion (see (2)) we suppose that all components can be complex (instead of real) numbers we arrive at the definition of complex quaternions (biquaternions).

Definition 2 *A complex quaternion (biquaternions) x is an object of the form*

$$x = \sum_{k=0}^{3} x_k e_k, \qquad x_k \in \mathbb{C}, \quad 0 \leq k \leq 3$$

with the commutation rule for the usual complex imaginary unit i with the quaternionic imaginary unit e_k, $k = 1, 2, 3$

$$ie_k = e_k i.$$

The algebra of complex quaternions will be denoted by $\mathbb{H}(\mathbb{C})$.

Now we consider functions f defined in Ω with values in $\mathbb{H}(\mathbb{C})$. Those functions may be written as

$$f(x) = \sum_{k=0}^{3} f_k(x)e_k, \quad f_k(x) \in \mathbb{C}, \ x = (x_1, x_2, x_3) \in \Omega. \tag{5}$$

Properties such as continuity, differentiability, integralbility, and so on, which are described to f have to be possessed by all components $f_k(x)$ which are complex-valued functions defined on Ω.

Let $\mathcal{B}(\Omega)$ be a function space of complex functions defined on Ω. For example, $\mathcal{B}$ may be C^k, $C^{(k,\varepsilon)}$, L_p, W_p^k and so on. We then define a function space

$$\begin{aligned}&\mathcal{B}(\Omega, \mathbb{H}(\mathbb{C})) :\\ &= \{f : \Omega \to \mathbb{H}(\mathbb{C}) : \text{ all the component of } f \text{ belong to } B(\Omega)\}.\end{aligned}$$

If $\mathcal{B}(\Omega)$ is normed with norm $||\cdot||_{\mathcal{B}}$ then we can define a norm on $\mathcal{B}(\Omega, \mathbb{H}(\mathbb{C}))$ by

$$||f||_{\mathcal{B}} = \big(\sum_{k=0}^{3} ||f_k||_{\mathcal{B}}^2\big)^{\frac{1}{2}} \text{ for } f \in \mathcal{B}(\Omega, \mathbb{H}(\mathbb{C})).$$

If $\mathcal{B}(\Omega)$ is a Banach space then the space $\mathcal{B}(\Omega, \mathbb{H}(\mathbb{C}))$ defined in this manner is also a complex Banach space. In this way the usual Banach spaces of these functions are denoted by $C^k(\Omega, \mathbb{H}(\mathbb{C}))$, $C^{(k,\varepsilon)}(\Omega, \mathbb{H}(\mathbb{C}))$, $L_p(\Omega, \mathbb{H}(\mathbb{C})), W_p^k(\Omega, \mathbb{H}(\mathbb{C}))$ and so on.

Let us consider $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ as a right module. Then, it can be easily verified that the formula

$$(f, g) := \int_{\Omega} \overline{f(x)} g(x) dx,$$

defines a scalar product turning $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ into a right Hilbert $\mathbb{H}(\mathbb{C})$-module.

For the definition as well of fractional order Sobolev spaces W_p^s, $s \in \mathbb{R}^+$ as of weighted L_2- spaces we refer the reader to [1, Chap. VII].

Definition 3 *Let $f \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$. The Moisil-Teodorescu differential operator is given by*

$$Df := \sum_{k=1}^{3} e_k \partial_k f \text{ where } \partial_k := \frac{\partial}{\partial x_k}.$$

If we write $f = f_0 + \vec{f}$ then we get by a straightforward calculation

$$Df = -\mathrm{div}\vec{f} + \mathrm{grad} f_0 + \mathrm{rot}\vec{f}. \tag{6}$$

Let us note that the *Moisil-Teodorescu operator* was introduced as acting from the left-hand side. The correponding operator acting from the right-hand side will be denoted by D_r

$$D_r f = \sum_{k=1}^{3} \partial_k f e_k$$

and in vector form the application of D_r can be represented as

$$D_r f = -\mathrm{div}\vec{f} + \mathrm{grad} f_0 - \mathrm{rot}\vec{f}. \tag{7}$$

As the Laplacian also the Helmholtz operator can be factorized in quaternionic analysis as

$$\Delta + \alpha^2 = -(D + \alpha)(D - \alpha). \tag{8}$$

Definition 4 *Let the operator $D_\alpha = D + \alpha I$ be given where α is an arbitrary complex constant and I is the identity operator.*

Hence, the equality (8) can be rewritten as

$$\Delta + \alpha^2 = -D_\alpha D_{-\alpha} = -D_{-\alpha} D_\alpha.$$

This means that any function satisfying the equation $D_\alpha f = 0$ or $D_{-\alpha} f = 0$ also satisfies the Helmholtz equation $(\Delta + \alpha^2) f = 0$.

We define

$$\mathcal{M}_p^k(\Omega, \mathbb{H}(\mathbb{C})) := \{f |\ f \in L_p^k(\Omega, \mathbb{H}(\mathbb{C})),\ D_\alpha^k f = 0\},$$

$$\mathcal{M}_{r,p}^k(\Omega, \mathbb{H}(\mathbb{C})) := \{f |\ f \in L_p^k(\Omega, \mathbb{H}(\mathbb{C})),\ D_{r,\alpha}^k f = 0\}.$$

In the case $p = 2,\ k = 1$ we write $\ker D_\alpha = \mathcal{M}_2^1(\Omega, \mathbb{H}(\mathbb{C}))$. Each element f belonging to $\ker D_\alpha$ is called a *left $\alpha-$hyperholomorphic*

function. Obviously, $\ker D_{r,\alpha}$ is the set of *right* $\alpha-$*hyperholomorphic functions.*

The subspace of *polymetaharmonic functions* of order k in $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ is denoted by

$$\mathbb{H}_{k,2} = \left\{ f,\ f \in L_2(\Omega, \mathbb{H}(\mathbb{C})),\ (\Delta + \alpha^2)^k f = 0 \text{ in } \Omega \right\}.$$

Now, using the equality (8) and the fundamental solution of the Helmholtz equation, a fundamental solution can be constructed for the factors of the Helmholtz operator. Indeed, if we assume ϑ to be a fundamental solution of the Helmholtz operator, i.e a function satisfying $(\Delta + \alpha^2)\vartheta(x) = \delta(x)$, where $\delta(x)$ is the Dirac delta distribution, then $K_\alpha(x) = -(D - \alpha)\vartheta(x)$ is a fundamental solution of D_α and $K_{-\alpha}(x) = -(D + \alpha)\vartheta(x)$ is a fundamental solution of $D_{-\alpha}$, i.e., $D_\alpha K_\alpha(x) = \delta(x)$, and $D_{-\alpha}K_{-\alpha}(x) = \delta(x)$.
As discussed in [13, p. 27] a unique fundamental solution to the Helmholtz operator related to its physical meaning is

$$\vartheta(x) = -\frac{e^{i\alpha|x|}}{4\pi|x|}.$$

Since $\vartheta(x)$ is a scalar function and using formulas (6), (7)we have $D_\alpha\vartheta(x) = D_{r,\alpha}\vartheta(x)$.

From formula (6) by a straightforward computation we get

$$K_\alpha(x) = -\text{grad}\vartheta(x) + \alpha\vartheta(x) = (\alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|})\ (-\frac{e^{i\alpha|x|}}{4\pi|x|}) \tag{9}$$

$$K_{-\alpha}(x) = -\text{grad}\vartheta(x) - \alpha\vartheta(x) = (-\alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|})\ (-\frac{e^{i\alpha|x|}}{4\pi|x|}) \tag{10}$$

Note that, by the definiton of the operators D_α and $D_{r,\alpha}$ we have

$$\overline{D_\alpha f(x)} = -D_{r,-\alpha}\overline{f(x)},\ \text{for any } \alpha \in \mathbb{C}. \tag{11}$$

Theorem 1 *(Quaternionic Stokes formula)*
Let f and g belong to $C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$ then

$$\int_\Omega [(D_r f(y))g(y) + f(y)Dg(y)]dy = \int_\Gamma f(y)\vec{n}(y)g(y)d\Gamma_y, \tag{12}$$

where $\vec{n} := \sum_{k=1}^{3} n_k e_k$ denotes the outward unitary normal on Γ.

The above theorem is taken from [13, Theorem 2], whose proof can be found in [9, Proposition 3.22].

Remark 1 From (12) with $\alpha \in \mathbb{C}$ we have

$$\int_\Omega [(D_{r,-\alpha} f(y))g(y) + f(y)(D_\alpha g(y))]dy = \int_\Gamma f(y)\vec{n}(y)g(y)d\Gamma_y, \quad (13)$$

$$\int_\Omega [(D_{r,\alpha} f(y))g(y) + f(y)(D_{-\alpha} g(y))]dy = \int_\Gamma f(y)\vec{n}(y)g(y)d\Gamma_y. \quad (14)$$

Theorem 2 *(Quaternionic Cauchy-Pompeiu formulas)*
Let $f \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$. *Then*

$$f(x) = -\int_\Gamma K_\alpha(x-y)\vec{n}(y)f(y)d\Gamma_y + \int_\Omega K_\alpha(x-y)D_\alpha f(y)dy. \quad (15)$$

It is straighforward to see

$$f(x) = -\int_\Gamma K_{-\alpha}(x-y)\vec{n}(y)f(y)d\Gamma_y + \int_\Omega K_{-\alpha}(x-y)D_{-\alpha} f(y)dy. \quad (16)$$

The formulas (15), (16) are the quaternionic Cauchy-Pompeiu formulas for the operators $D_\alpha,\ D_{-\alpha}$. Let us now introduce the main integral operators whose properties are similar to their famous complex prototypes. The *Cauchy's integral operator,* the first order *Teodorescu transform* operator and the operator of *singular integration,* guarantee an efficient solution of different kinds of boundary value problems.

$$\begin{aligned} F_\alpha f(x) &= -\int_\Gamma K_\alpha(x-y)\vec{n}(y)f(y)d\Gamma_y, \quad x \in \mathbb{R}^3 - \Gamma, \\ T_{\alpha,1} f(x) &= \int_\Omega K_\alpha(x-y)f(y)dy, \quad x \in \mathbb{R}^3, \\ S_\alpha f(x) &= -2\int_\Gamma K_\alpha(x-y)\vec{n}(y)f(y)d\Gamma_y, \quad x \in \Gamma. \end{aligned}$$

Note that the integral in the definition of the operator S_α is taken in the sense of Cauchy's principal value. In the other words, it can be called a singular Cauchy operator.

As usual, the singular Cauchy operator generates two important operators P_α and Q_α. Which are defined by

$$\begin{cases} P_\alpha &= \frac{1}{2}(I + S_\alpha), \\ Q_\alpha &= \frac{1}{2}(I - S_\alpha). \end{cases} \quad (17)$$

Theorem 3 *(Quaternionic Plemelj-Sokhotski formulas)*
Let Γ *be a closed Liapunov surface,* $f \in C^{0,\varepsilon}(\Gamma, \mathbb{H}(\mathbb{C}))$, $0 < \varepsilon \leq 1$. *Then everywhere on* Γ *the following limits exist satisfying*

$$\lim_{\Omega^+ \ni x \to \tau \in \Gamma} F_\alpha f(x) = P_\alpha f(\tau),$$

$$\lim_{\Omega^- \ni x \to \tau \in \Gamma} F_\alpha f(x) = -Q_\alpha f(\tau),$$

where $\Omega^+ = \Omega$, $\Omega^- := \mathbb{R}^3 - \overline{\Omega}$.

In the book [8, Theorem 2.5.10], [9, Theorem 3.64], the authors have proven the theorem in the case $\alpha = 0$. For the case $\alpha \neq 0$ the proof can be found in [13, Theorem 8]. Some of the following properties of the operators P_α, Q_α and S_α are immediate consequences of Theorem 3 and the Cauchy-Pompeiu formula.

Theorem 4 *Let* $f \in C^{0,\varepsilon}(\Gamma, \mathbb{H}(\mathbb{C}))$, $0 < \varepsilon \leq 1$. *Then the equations*

(i) $(S_\alpha^2 u)(\tau) = u(\tau)$,

(ii) $(F_\alpha P_\alpha u)(\tau) = (F_\alpha u)(\tau)$,

(iii) $(P_\alpha^2 u)(\tau) = (P_\alpha u)(\tau)$,

(iv) $(Q_\alpha^2 u)(\tau) = (Q_\alpha u)(\tau)$,

are valid for any $\tau \in \Gamma$

We refer to [13, Chap. 2, Theorem 9] and [8, Corollary 4.2.7] for its proofs. The operator S_α is an involution on the space $C^{0,\varepsilon}(\Gamma, \mathbb{H}(\mathbb{C}))$, $0 < \varepsilon \leq 1$, hence, P_α, Q_α are mutually complementary projection operators on the same space.

3 Integral representations for metaharmonic functions

As introduced in the former section, all *left (right)* $\alpha-$*hyperholomorphic functions* are *metaharmonic functions* in all their coordinates. Therefore, it seems to be necessary to consider the links between *left (right)* $\alpha-$*hyperholomorphic functions* and *metaharmonic functions*. We begin with an integral representation formula for arbitrary C^2- functions. This formula can be obtained by iterating both formulas (15) and (16).

Theorem 5 *Let $f \in C^2(\Omega, \mathbb{H}(\mathbb{C})) \cap C^1(\overline{\Omega}, \mathbb{H}(\mathbb{C}))$. Then*

$$\begin{aligned} f(x) &= -\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)f(y)d\Gamma_y + \int_\Gamma \vartheta(x-y)\vec{n}(y)D_{\alpha,y}f(y)d\Gamma_y \\ &\quad - \int_\Omega \vartheta(x-y)D_{-\alpha,y}D_{\alpha,y}f(y)dy, \end{aligned} \tag{18}$$

where $K_\alpha^{(1)}(x) := K_\alpha(x)$.

Proof Applying the quaternionic Cauchy-Pompeiu formula (16) for $D_{\alpha,y}f(y)$ we obtain

$$\begin{aligned} D_{\alpha,y}f(y) &= -\int_\Gamma K_{-\alpha}^{(1)}(y-\tilde{y})\vec{n}(\tilde{y})D_{\alpha,\tilde{y}}f(\tilde{y})d\Gamma_{\tilde{y}} \\ &\quad + \int_\Omega K_{-\alpha}^{(1)}(y-\tilde{y})D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y}. \end{aligned}$$

This leads to

$$\begin{aligned} f(x) = &-\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)f(y)d\Gamma_y \\ &-\int_\Gamma \psi_(x,\tilde{y})\vec{n}(\tilde{y})D_{\alpha,\tilde{y}}f(\tilde{y})d\Gamma_y + \int_\Omega \psi(x,\tilde{y})D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y}, \end{aligned}$$

where $\psi(x,\tilde{y}) = \int_\Gamma K_\alpha^{(1)}(x-y)K_{-\alpha}^{(1)}(y-\tilde{y})dy$.
For $x, \tilde{y} \in \Omega$ with $x \neq \tilde{y}$ formula (15) yields $-\vartheta(x-\tilde{y}) = \psi(x,\tilde{y}) + \tilde{\psi}(x,\tilde{y})$ where

$$\tilde{\psi}(x,\tilde{y}) = \int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)\vartheta(y-\tilde{y})d\Gamma_y$$

Since $\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)K_\alpha^{(1)}(y-\tilde{y})d\Gamma_y = 0$, we have $(D_{r,\tilde{y}} + \alpha)\tilde{\psi}(x,\tilde{y}) = 0$ and for $x \neq \tilde{y}$, the equality (13) now yields

$$\int_\Gamma \tilde{\psi}_(x,\tilde{y})\vec{n}(\tilde{y})D_{\alpha,\tilde{y}}f(\tilde{y})d\Gamma_y - \int_\Omega \tilde{\psi}(x,\tilde{y})D_{-\alpha,\tilde{y}}D_{\alpha,\tilde{y}}f(\tilde{y})d\tilde{y} = 0.$$

This leads to

$$\begin{aligned} f(x) = &-\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)f(y)d\Gamma_y \\ &+\int_\Gamma \vartheta(x-y)\vec{n}(y)D_{\alpha,y}f(y)dy - \int_\Omega \vartheta(x-y)D_{-\alpha,y}D_{\alpha,y}f(y)dy \end{aligned}$$

From this desired expression we get immediately the following results.

Corollary 1 *Let $f \in C^2(\Omega, \mathbb{H}(\mathbb{C})) \cap C^1(\overline{\Omega}, \mathbb{H}(\mathbb{C}))$ be a metaharmonic function. Then*

$$f(x) = -\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)f(y)d\Gamma_y + \int_\Gamma \vartheta(x-y)\vec{n}(y)D_{\alpha,y}f(y)d\Gamma_y.$$

This corollary shows that a metaharmonic function allows a simple integral representation on the basis of its boundary values $f|_\Gamma$ and $D_\alpha|_\Gamma$.

Corollary 2 *If $f \in C^2(\Omega, \mathbb{H}(\mathbb{C})) \cap C^1(\overline{\Omega}, \mathbb{H}(\mathbb{C}))$ is $\alpha-$ hyperholomorphic, then*

$$f(x) = -\int_\Gamma K_\alpha^{(1)}(x-y)\vec{n}(y)f(y)d\Gamma_y.$$

This means that, from our integral representation we get directly the Cauchy formula for $\alpha-$*hyperholomorphic functions.*

We define an analogue of the Newtonian potential

$$T_{\Delta+\alpha^2}f(x) = \int_\Omega \vartheta(x-y)f(y)dy$$

where $\vartheta(x-y) = -\frac{e^{i\alpha|x-y|}}{4\pi|x-y|},\ f \in L_1(\Omega, \mathbb{H}(\mathbb{C})),\ x \in \Omega$.
Analogously, the acoustic single layer potential is defined by

$$V_\alpha f(x) = \int_\Gamma \vartheta(x-y)\vec{n}(y)D_{\alpha,y}f(y)d\Gamma_y.$$

By the same method as in [11] we can prove that V_α is a bounded operator from $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ into $W_2^2(\Omega, \mathbb{H}(\mathbb{C}))$ and $T_{\Delta+\alpha^2}$ is defined from $C(\overline{\Omega}, \mathbb{H}(\mathbb{C}))$ to $C(\overline{\Omega}, \mathbb{H}(\mathbb{C}))$.

Noting that $D_\alpha\vartheta(x) = -K_{-\alpha}(x),\ D_{-\alpha}\vartheta(x) = -K_\alpha(x)$ and $\vartheta(x)$ has a singularity of order 1, and using methods as in [11, Theorem 6.4] as well as the formula (6), by a straightforward calculation

$$\begin{aligned} D_\alpha T_{\Delta+\alpha^2}f(x) &= -T_{-\alpha,1}f(x), \quad D_{-\alpha}T_{\Delta+\alpha^2}f(x) = -T_{\alpha,1}f(x), \\ D_\alpha V_\alpha f(x) &= -F_{-\alpha}f(x), \quad D_{-\alpha}V_\alpha f(x) = -F_\alpha f(x), \end{aligned} \tag{19}$$

follow. Moreover, applying the Stokes formula (13) we get

$$\begin{aligned} &\int_\Omega \left[D_{r,-\alpha,y}\vartheta(x-y)f(y) + \vartheta(x-y)D_{\alpha,y}f(y)\right]dy \\ &= \int_\Gamma \vartheta(x-y)\vec{n}(y)f(y)d\Gamma_y. \end{aligned}$$

This means that

$$V_\alpha f(x) = T_{-\alpha,1} f(x) + T_{\Delta+\alpha^2} D_\alpha f(x). \tag{20}$$

It is easily seen that

$$V_\alpha f(x) = T_{\alpha,1} f(x) + T_{\Delta+\alpha^2} D_{-\alpha} f(x), \tag{21}$$

if we apply the Stokes formula (14). The equations (19), (20) and (21) are valid in $C^\varepsilon(\Omega, \mathbb{H}(\mathbb{C}))$ and in Sobolev spaces $W_p^k(\Omega, \mathbb{H}(\mathbb{C}))$. The equations (20) and (21) connect operators from the classical potential theory with operators arising in complex quaternionic analysis.

Remark 1 *Let us introduce the boundary operator*

$$F_{\Delta+\alpha^2} f(x) = F_\alpha f(x) + V_\alpha f(x).$$

Then the Cauchy-Pompeiu formula (18) for the term of Helmholtz equation can be rewritten as

$$f(x) = F_{\Delta+\alpha^2} Tr_\Gamma f(x) + T_{\Delta+\alpha^2}(\Delta + \alpha^2) f(x)$$

where

$$Tr_\Gamma f(x) = \begin{pmatrix} tr_\Gamma f \\ tr_\Gamma D_\alpha f \end{pmatrix}.$$

The following integral representation formulas in terms of powers of the Helmholtz equation are developed in [11].

Theorem 6 *Let $f \in C^{2n}(\Omega, \mathbb{H}(\mathbb{C})) \cap C^{2n-1}(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$. Then*

$$\begin{aligned} f(x) &= \sum_{k=1}^{n} \Big[- \int_\Gamma D_{\alpha,y} \vartheta^{(k)}(x-y) \vec{n}(y) (D_{-\alpha,y} D_{\alpha,y})^k f(y) d\Gamma_y \\ &\quad + \int_\Gamma \vartheta^{(k)}(x-y) \vec{n}(y) D_{\alpha,y} (D_{-\alpha,y} D_{\alpha,y})^k f(y) d\Gamma_y \Big] \\ &\quad - \int_\Omega \vartheta^{(n)}(x-y) (D_{-\alpha,y} D_{\alpha,y})^n f(y) dy \end{aligned} \tag{22}$$

where

$$\begin{aligned} \vartheta^{(1)}(x) &= \vartheta(x), \\ \vartheta^{(2)}(x) &= \frac{1}{2\alpha} \frac{\partial}{\partial \alpha} \vartheta^{(1)}(x), \\ \vartheta^{(k+1)}(x) &= \frac{1}{2k\alpha} \frac{\partial}{\partial \alpha} \vartheta^{(k)}(x) \end{aligned}$$

4 Orthogonal decomposition of $L_2(\Omega, \mathbb{H}(\mathbb{C}))$

We start with a decomposition of the space $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ with respect to *metaharmonic functions.*

Theorem 7 *The space $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ allows the orthogonal decomposition*

$$\begin{aligned} L_2(\Omega, \mathbb{H}(\mathbb{C})) &= \ker(\Delta + \alpha^2) \cap L_2(\Omega, \mathbb{H}(\mathbb{C})) \oplus \\ &\quad \left[(\Delta + \alpha^2)\{W_2^2(\Omega, \mathbb{H}(\mathbb{C}))\} \cap \ker tr_\Gamma \cap \ker tr_\Gamma D_\alpha\right]. \end{aligned} \tag{23}$$

Proof Notice that for $u, \upsilon \in L_2(\Omega, \mathbb{H}(\mathbb{C}))$ we have

$$(D_\alpha u, \upsilon) = \int\limits_\Omega \overline{D_\alpha u(x)}\upsilon(x)dx = -\int\limits_\Omega D_{r,-\alpha}\overline{u(x)}\upsilon(x)dx.$$

Using the Stokes formula (13) we have

$$\int\limits_\Gamma \overline{u(x)}\vec{n}(x)\upsilon(x)d\Gamma_x = \int\limits_\Omega D_{r,-\alpha}\overline{u(x)}\upsilon(x)dx + \int\limits_\Omega \overline{u(x)}D_\alpha\upsilon(x)dx.$$

This means that

$$(D_\alpha u, \upsilon) - (u, D_\alpha\upsilon) = -\int\limits_\Gamma \overline{u(x)}\vec{n}(x)\upsilon(x)d\Gamma_x.$$

If we use the Stokes formula (14) we also get

$$(D_{-\alpha} u, \upsilon) - (u, D_{-\alpha}\upsilon) = -\int\limits_\Gamma \overline{u(x)}\vec{n}(x)\upsilon(x)d\Gamma_x.$$

Now we look at

$$\begin{aligned} &((\Delta + \alpha^2)u, \upsilon) = -(D_\alpha\{D_{-\alpha}u\}, \upsilon) \\ &= \int\limits_\Gamma \overline{D_{-\alpha}u(x)}\vec{n}(x)\upsilon(x)d\Gamma_x - (D_{-\alpha}u, D_\alpha\upsilon) \\ &= \int\limits_\Gamma \overline{D_{-\alpha}u(x)}\vec{n}(x)\upsilon(x)d\Gamma_x + \int\limits_\Gamma \overline{u(x)}\vec{n}(x)D_\alpha\upsilon(x)d\Gamma_x - (u, D_{-\alpha}D_\alpha\upsilon) \end{aligned}$$

$$= \int_\Gamma \overline{D_{-\alpha}u(x)}\vec{n}(x)\upsilon(x)d\Gamma_x + \int_\Gamma \overline{u(x)}\vec{n}(x)D_\alpha\upsilon(x)d\Gamma_x + (u,(\Delta+\alpha^2)\upsilon).$$

Therefore, we obtain

$$\begin{aligned}((\Delta+\alpha^2)u,\upsilon) &= \int_\Gamma \overline{D_\alpha u(x)}\vec{n}(x)\upsilon(x)d\Gamma_x \\ &\quad + \int_\Gamma \overline{u(x)}\vec{n}(x)D_{-\alpha}\upsilon(x)d\Gamma_x + (u,(\Delta+\alpha^2)\upsilon).\end{aligned}$$

The above formula shows that the subspaces $\ker(\Delta+\alpha^2)\cap L_2(\Omega,\mathbb{H}(\mathbb{C}))$ and

$$\left[(\Delta+\alpha^2)\{W_2^2(\Omega,\mathbb{H}(\mathbb{C}))\}\cap\ker tr_\Gamma\cap\ker tr_\Gamma D_\alpha\right]$$

are orthogonal subspaces.
Notice that, if we define

$$\overset{0}{W}{}^2_{2,\Delta+\alpha^2}(\Omega,\mathbb{H}(\mathbb{C})) := \Big\{f,\ f\in W^2_{2,\Delta+\alpha^2}(\Omega,\mathbb{H}(\mathbb{C})),\ tr_\Gamma f=0,\ tr_\Gamma D_\alpha f=0\ \Big\},$$

then the formula (23) can be rewritten as

$$\begin{aligned}&L_2(\Omega,\mathbb{H}(\mathbb{C}))\\ &= \ker(\Delta+\alpha^2)\cap L_2(\Omega,\mathbb{H}(\mathbb{C}))\oplus(\Delta+\alpha^2)(\overset{0}{W}{}^2_{2,\Delta+\alpha^2}(\Omega,\mathbb{H}(\mathbb{C}))).\end{aligned}$$

Now, we can denote the corresponding orthoprojections by

$$\begin{aligned}P_{\Delta+\alpha^2}:\ L_2(\Omega,\mathbb{H}(\mathbb{C})) &\longrightarrow \ker(\Delta+\alpha^2)\cap L_2(\Omega,\mathbb{H}(\mathbb{C}))\\ Q_{\Delta+\alpha^2}:\ L_2(\Omega,\mathbb{H}(\mathbb{C})) &\longrightarrow (\Delta+\alpha^2)(\overset{0}{W}{}^2_{2,\Delta+\alpha^2}(\Omega,\mathbb{H}(\mathbb{C}))).\end{aligned}$$

The following theorem can be considered as an extension of the results in [2] for classical *polymetaharmonic functions.*

Theorem 8 *The space $L_2(\Omega,\mathbb{H}(\mathbb{C}))$ allows the orthogonal decomposition*

(i)

$$L_2(\Omega,\mathbb{H}(\mathbb{C})) = \mathcal{M}_2^k(\Omega,\mathbb{H}(\mathbb{C}))\oplus D_\alpha^k(\overset{0}{W}{}^k_{2,\alpha}(\Omega,\mathbb{H}(\mathbb{C}))),$$

where

$$\begin{aligned}\overset{0}{W}{}^k_{2,\alpha}(\Omega,\mathbb{H}(\mathbb{C})) :&= \{f,\ f\in L_2(\Omega,\mathbb{H}(\mathbb{C}),\ D_\alpha^\nu f=0 \text{ on } \Gamma\\ &\qquad \text{for } 0\le\nu\le \mathrm{k}-1\},\end{aligned}$$

(ii)

$$L_2(\Omega, \mathbb{H}(\mathbb{C})) = \mathbb{H}_2^k(\Omega, \mathbb{H}(\mathbb{C})) \oplus (\Delta + \alpha^2)^k (\overset{0}{W}{}^k_{2,\Delta+\alpha^2}(\Omega, \mathbb{H}(\mathbb{C})),$$

where

$$\begin{aligned}\overset{0}{W}{}^k_{2,\Delta+\alpha^2}(\Omega, \mathbb{H}(\mathbb{C})) := \{f,\ f \in L_2(\Omega, \mathbb{H}(\mathbb{C})),\ (\Delta+\alpha^2)^\nu f = 0,\\ D_\alpha(\Delta+\alpha^2)^\nu f = 0 \text{ on } \Gamma \text{ for } 0 \le \nu \le k-1\}.\end{aligned}$$

Proof (i) For $u, v \in L_2(\Omega, \mathbb{H}(\mathbb{C}))$ and using the Stokes formula (13) we have

$$\begin{aligned}(D_\alpha^k u, v) &= -\int_\Gamma \overline{D_\alpha^{k-1} u(x)} \vec{n}(x) v(x) d\Gamma_x + (D_\alpha^{k-1} u, D_\alpha v)\\ &= -\int_\Gamma \overline{D_\alpha^{k-1} u(x)} \vec{n}(x) v(x) d\Gamma_x\\ &\quad -\int_\Gamma \overline{D_\alpha^{k-2} u(x)} \vec{n}(x) D_\alpha v(x) d\Gamma_x + (D_\alpha^{k-2} u, D_\alpha v)\\ &= \cdots\\ &= -\int_\Gamma \overline{D_\alpha^{k-1} u(x)} \vec{n}(x) v(x) d\Gamma_x - \int_\Gamma \overline{D_\alpha^{k-2} u(x)} \vec{n}(x) D_\alpha v(x) d\Gamma_x\\ &\quad -\cdots - \int_\Gamma \overline{D_\alpha u(x)} \vec{n}(x) D_\alpha^{n-2} v(x) d\Gamma_x\\ &\quad -\int_\Gamma \overline{u(x)} \vec{n}(x) D_\alpha^{n-1} v(x) d\Gamma_x + (u, D_\alpha^n v).\end{aligned}$$

The above equality shows that the inner product of each element $v \in \overset{0}{W}{}^k_{2,\alpha}(\Omega,\ \mathbb{H}(\mathbb{C}))$ and any $u \in L_2(\Omega, \mathbb{H}(\mathbb{C}))$ equal zero if and only if $u \in \mathcal{M}_2^k(\Omega, \mathbb{H}(\mathbb{C}))$. This means that the subspaces $\mathcal{M}_2^k(\Omega, \mathbb{H}(\mathbb{C}))$ and $\overset{0}{W}{}^k_{2,\alpha}(\Omega, \mathbb{H}(\mathbb{C}))$ are orthogonal subspaces.

(ii)By the same method as in the above proofs of the assertion (i) and Theorem 7, the assertion (ii) of the theorem now follows.

Here, the more general decompositions of the complex quaternion valued bimodul-Hilbert space open the door for the consideration of further classes of boundary value problems of partial differential equations.

5 Dirichlet problem for bimetaharmonic functions

In this section, we study the bimetaharmonic problem by the help of the self-contained theory given in the preceding sections. Questions of existence of solutions are answered in this section as well as an important convenient representation of the solutions. We hope these integral representation formulas will be adapted to the necessary numerical evaluation of the solutions which can be connected with mathematical models. For that purpose, we first assert the origin of the projections onto *metaharmonic functions.*

Looking at the formula (22), each bimetaharmonic function, i.e., satistyfying the equation $(\Delta+\alpha^2)^2 f(x)=0$, has the integral representation

$$\begin{aligned} f(x) &= -\int_\Gamma D_{\alpha,y}\vartheta^{(1)}(x-y)\vec{n}(y)f(y)d\Gamma_y \\ &\quad +\int_\Gamma \vartheta^{(1)}(x-y)\vec{n}(y)D_{\alpha,y}f(y)d\Gamma_y \\ &\quad +\int_\Gamma D_{\alpha,y}\vartheta^{(2)}(x-y)\vec{n}(y)(\Delta+\alpha^2)f(y)d\Gamma_y \\ &\quad +\int_\Gamma \vartheta^{(2)}(x-y)\vec{n}(y)D_{\alpha,y}(\Delta+\alpha^2)f(y)d\Gamma_y \\ &= F_\alpha f(x)+V_\alpha D_\alpha f(x) \\ &\quad +\int_\Gamma D_{\alpha,y}\vartheta^{(2)}(x-y)\vec{n}(y)(\Delta+\alpha^2)f(y)d\Gamma_y \\ &\quad -\int_\Gamma \vartheta^{(2)}(x-y)\vec{n}(y)D_{\alpha,y}(\Delta+\alpha^2)f(y)d\Gamma_y. \end{aligned}$$

Applying formula (22) for $D_\alpha f$ gives

$$\begin{aligned} D_\alpha f(x) &= F_{-\alpha}(D_\alpha f(x))-V_\alpha(\Delta+\alpha^2)f(x) \\ &\quad +\int_\Gamma D_{-\alpha,y}\vartheta^{(2)}(x-y)\vec{n}(y)D_{\alpha,y}(\Delta+\alpha^2)f(y)d\Gamma_y. \end{aligned}$$

If we take the traces of f and $D_\alpha f$ then we obtain the following boundary integral equation, $x_0\in\Gamma$,

$$f(x_0)= \frac{1}{2}\left(f(x_0)+S_\alpha f(x_0)\right)+V_\alpha D_\alpha f(x_0)$$

$$
\begin{aligned}
&+\int_\Gamma D_{\alpha,y}\vartheta^{(2)}(x_0-y)\vec{n}(y)(\Delta+\alpha^2)f(y)d\Gamma_y\\
&-\int_\Gamma \vartheta^{(2)}(x_0-y)\vec{n}(y)D_{\alpha,y}(\Delta+\alpha^2)f(y)d\Gamma_y,\\
D_\alpha f(x_0) = \quad &\frac{1}{2}\left(D_\alpha f(x_0)+S_{-\alpha}D_\alpha f(x_0)\right)-V_\alpha(\Delta+\alpha^2)f(x_0)\\
&+\int_\Gamma D_{-\alpha,y}\vartheta^{(2)}(x_0-y)\vec{n}(y)D_{\alpha,y}(\Delta+\alpha^2)f(y)d\Gamma_y.
\end{aligned}
$$

If we now are looking for all bimetaharmonic function u, which are also metaharmonic functions then we get the following conditions

$$
\begin{aligned}
f(x_0) &= \frac{1}{2}\left(f(x_0)+S_\alpha f(x_0)\right)+V_\alpha D_\alpha f(x_0),\\
D_\alpha f(x_0) &= \frac{1}{2}\left(D_\alpha f(x_0)+S_{-\alpha}D_\alpha f(x_0)\right),
\end{aligned}
$$

or with orther words

$$
\begin{pmatrix} f \\ D_\alpha f \end{pmatrix} = \begin{pmatrix} P_\alpha & V_\alpha \\ 0 & P_{-\alpha} \end{pmatrix}\begin{pmatrix} f \\ D_\alpha f \end{pmatrix}.
$$

From this we can derive the projections

$$
P_{\Delta+\alpha^2} = \begin{pmatrix} P_\alpha & V_\alpha \\ 0 & P_{-\alpha} \end{pmatrix}, \quad Q_{\Delta+\alpha^2} = \begin{pmatrix} Q_\alpha & -V_\alpha \\ 0 & Q_{-\alpha} \end{pmatrix}.
$$

These operators as well as the orthogonal decomposition of $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ given in the preceding section will open the door for the consideration of further classes of boundary value problems for bimetaharmonic functions.

We now come to rewrite the form of the Dirichlet problem for the bimetaharmonic equation for sake of convenience. With the notation

$$
Tr_\Gamma f(x) = \begin{pmatrix} tr_\Gamma f \\ tr_\Gamma D_\alpha f \end{pmatrix},
$$

then the Dirichlet problem for the inhomogeneous bimetaharmonic equation can be written as

$$
\begin{cases} (\Delta+\alpha^2)^2 u &= f \quad \text{in} \quad \Omega, \\ Tr_\Gamma u &= g \quad \text{on} \quad \Gamma. \end{cases} \tag{24}
$$

By [7, Theorem 4] (see also [9, Theorem 4.57]) we are able to state the following theorem.

Theorem 9 *Suppose that Ω is a domain with sufficiently smooth boundary Γ. For each pair of functions $\omega_1 \in W_2^{k+3/2}(\Gamma, \mathbb{H}(\mathbb{C}))$, $\omega_2 \in W_2^{k+1/2}(\Gamma, \mathbb{H}(\mathbb{C}))$ there exists an extension $h \in W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))$ with $h|_\Gamma = \omega_1$, and $(D+\alpha)h|_\Gamma = \omega_2$, where α is a complex constant.*

Proof By [7, Theorem 4], there exists a $W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))-$extension h such that $h|_\Gamma = \omega_1$, and $Dh|_\Gamma = \omega_2 - \alpha\omega_1$. This yields Theorem 9.

We now can look for solutions of the problem (24). To do this, we start with the following problem

$$\begin{cases} (\Delta+\alpha^2)^2 u &= 0 \quad \text{in} \quad \Omega, \\ Tr_\Gamma u &= g \quad \text{on} \quad \Gamma. \end{cases} \tag{25}$$

Theorem 10 *If $k \in \mathbb{N}$, $g \in W_2^{k+3/2}(\Gamma, \mathbb{H}(\mathbb{C})) \times W_2^{k+1/2}(\Gamma, \mathbb{H}(\mathbb{C}))$ then the problem (25) has a solution $u \in W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))$, which may be written as*

$$u = F_{\Delta+\alpha^2} g + T_{\Delta+\alpha^2} P_{\Delta+\alpha^2} (\Delta+\alpha^2) h$$

where h is the $W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))-$extension of g appearing in Theorem 9.

Proof Using Theorem 9, if $g \in W_2^{k+3/2}(\Gamma, \mathbb{H}(\mathbb{C})) \times W_2^{k+1/2}(\Gamma, \mathbb{H}(\mathbb{C}))$ then there exists an $h \in W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))$ such that $Tr_\Gamma h = g$. Let $u = v + h$ then our boundary value problem (25) has the form

$$\begin{cases} (\Delta+\alpha^2)^2 v &= -(\Delta+\alpha^2)^2 h \quad \text{in} \quad \Omega, \\ Tr_\Gamma v &= 0 \qquad\qquad\quad \text{on} \quad \Gamma. \end{cases} \tag{26}$$

We now are looking for solutions of the problem (26). Noting that $\vartheta^{(1)}$ has a singularity of order 1 and using [11, Theorem 6.4] we get $(\Delta+\alpha^2) T_{\Delta+\alpha^2} = I$ where I is the identity operator and

$$T_{\Delta+\alpha^2} \ : \ W_2^k(\Gamma, \mathbb{H}(\mathbb{C})) \longrightarrow W_2^{k+2}(\Gamma, \mathbb{H}(\mathbb{C})).$$

Moreover, the validity of Theorem 7 and the corresponding orthoprojection $Q_{\Delta+\alpha^2}$ show that there exists an $\mathbb{H}(\mathbb{C})-$valued function $v \in \overset{0}{W}{}_{2,\Delta+\alpha^2}^{2}(\Omega, \mathbb{H}(\mathbb{C}))$ such that

$$(\Delta+\alpha^2) v = -Q_{\Delta+\alpha^2} T_{\Delta+\alpha^2} (\Delta+\alpha^2)^2 h.$$

Using the Cauchy-Pompeiu formula for the Helmholtz equation in Remark 1, by the same methods as in [8, Theorem 4.2.14] for $\upsilon \in W^2_{2,\Delta+\alpha^2}(\Omega, \mathbb{H}(\mathbb{C}))$ and $Tr_\Gamma \upsilon = 0$ we obtain

$$\upsilon = -T_{\Delta+\alpha^2} Q_{\Delta+\alpha^2} T_{\Delta+\alpha^2} (\Delta+\alpha^2)^2 h.$$

Therefore,

$$\begin{aligned} (\Delta+\alpha^2)^2 \upsilon &= -(\Delta+\alpha^2)^2 T_{\Delta+\alpha^2} Q_{\Delta+\alpha^2} T_{\Delta+\alpha^2} (\Delta+\alpha^2)^2 h \\ &= -(\Delta+\alpha^2) Q_{\Delta+\alpha^2} T_{\Delta+\alpha^2} (\Delta+\alpha^2)^2 h \\ &= -(\Delta+\alpha^2)(I - P_{\Delta+\alpha^2}) T_{\Delta+\alpha^2} (\Delta+\alpha^2)^2 h \\ &= -(\Delta+\alpha^2)^2 h, \end{aligned}$$

because of the definition of the orthoprojection $P_{\Delta+\alpha^2}$, $\mathrm{Im} P_{\Delta+\alpha^2} \subset \ker(\Delta+\alpha^2)$.
Hence, using again the Cauchy-Pompeiu type formula for the Helmholtz equation we get

$$\begin{aligned} \upsilon &= -T_{\Delta+\alpha^2} Q_{\Delta+\alpha^2} \left[(\Delta+\alpha^2) h - F_{\Delta+\alpha^2} (\Delta+\alpha^2) h \right] \\ &= -T_{\Delta+\alpha^2} Q_{\Delta+\alpha^2} (\Delta+\alpha^2) h + T_{\Delta+\alpha^2} Q_{\Delta+\alpha^2} F_{\Delta+\alpha^2} (\Delta+\alpha^2) h. \end{aligned}$$

On the other hand, by the orthoprojections $\mathrm{Im} P_{\Delta+\alpha^2} = \ker Q_{\Delta+\alpha^2}$, and because $F_{\Delta+\alpha^2}$ maps onto $\mathrm{Im} P_{\Delta+\alpha^2}$ then we get $Q_{\Delta+\alpha^2} F_{\Delta+\alpha^2} (\Delta+\alpha^2) h = 0$.
Thus,

$$\begin{aligned} \upsilon &= -T_{\Delta+\alpha^2} Q_{\Delta+\alpha^2} (\Delta+\alpha^2) h \\ &= -T_{\Delta+\alpha^2} (\Delta+\alpha^2) h + T_{\Delta+\alpha^2} P_{\Delta+\alpha^2} (\Delta+\alpha^2) h. \end{aligned}$$

Meanwhile, the Cauchy-Pompeiu representation for the Helmholtz equation shows that

$$T_{\Delta+\alpha^2} (\Delta+\alpha^2) h = h - F_{\Delta+\alpha^2} h.$$

This leads to

$$\upsilon = -h + F_{\Delta+\alpha^2} h + T_{\Delta+\alpha^2} P_{\Delta+\alpha^2} (\Delta+\alpha^2) h.$$

Consequently, it may be observed that

$$\upsilon + h = F_{\Delta+\alpha^2} h + T_{\Delta+\alpha^2} P_{\Delta+\alpha^2} (\Delta+\alpha^2) h.$$

Hence,$u = F_{\Delta+\alpha^2} h + T_{\Delta+\alpha^2} P_{\Delta+\alpha^2} (\Delta+\alpha^2) h$.

Combining the problem (25) with (26) we obtain immediately the following proposition.

Proposition *Under the assumption of the above theorem, the problem (24) has a solution of the form*

$$u = F_{\Delta+\alpha^2} g + T_{\Delta+\alpha^2} P_{\Delta+\alpha^2} (\Delta + \alpha^2) h + T_{\Delta+\alpha^2} Q_{\Delta+\alpha^2} T_{\Delta+\alpha^2} f$$

where h denotes a $W_2^{k+2}(\Omega, \mathbb{H}(\mathbb{C}))-$extension of g.

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Cauchy-Pompeiu representation formulas in Clifford analysis

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Abstract

A general Cauchy-Pompeiu representation formula for Clifford-algebra valued function in regular domains of $\mathbb{R}^{m+1}$ related to the differential operator $\partial^\ell \Delta^k$ for $0 \leq \ell, 0 \leq k$ is developed where in the case of m odd k and ℓ are restricted to $0 \leq 2(k + [(\ell + 1)/2]) < m + 1$. This formula generalizes results from [3, 4] where only the first few ℓ were admitted. A fundamental solution to this differential operator is constructed which also leads to a higher order Pompeiu operator related to this differential operator.

Classification: 30G30, 31B10, 31B30

Keywords: Clifford analysis, higher order partial differential operators, fundamental solutions, integral representations, higher order Cauchy-Pompeiu formulas

1 Introduction

Cauchy-Pompeiu representation formulas are important for treating partial differential equations. In Clifford analysis the basic differential operator is the Dirac operator ∂ and its conjugate $\bar{\partial}$. The representation formula developed here are related to the general differential operator $\partial^k \bar{\partial}^\ell$ for $k, \ell \in \mathbb{N}_0, 0 < k + \ell$. For convenience instead $\partial^\ell \Delta^k$ is considered where $\Delta = \partial\bar{\partial} = \bar{\partial}\partial$ is the Laplace operator. In the particular case of complex analysis the representation formulas were given in [4], see also [1]. For the general concept of Clifford analysis the references [5, 6, 7] are recommended.

Definition 1 *Let $\{e_0, e_1, \ldots, e_m\}$ be an orthonormal basis of $\mathbb{R}^{m+1}$ and $\mathcal{A}$ be the 2^m-dimensional Eucledian space with the basis*

$$\mathcal{B} = \big\{ e_A = e_{h_1,h_2,\ldots,h_r} : \ A = \{h_1, h_2 \ldots, h_r\} \\ \subset \{0, \ldots, m\}, 1 \le h_1 < \ldots < h_r \le m \big\}$$

and $e_\emptyset = e_0 = 1$. With $e_{h_1,h_2,\ldots,h_r} = e_{h_1} e_{h_2} \ldots e_{h_r}$, $e_k e_\ell = -e_\ell e_k$, $e_k e_k = -1$ for $1 \le k < \ell \le m$ a multiplication is induced, leading to the (special) Clifford algebra $\mathcal{A}$ of elements $a = \sum_A a_A e_A$ with $a_A \in \mathbb{C}$.

For $m = 1$ the algebra $\mathcal{A}$ is identical to $\mathbb{C}$, so in the sequal $m \ge 2$ is assumed.

$\mathcal{A}$ is not commutative for $m \ge 2$.

For $m = 2$ the algebra $\mathcal{A}$ coincides with the quaternionic algebra.

Definition 2

1. *By $\overline{e_0} = e_0$, $\overline{e_k} = -e_k$ $(1 \le k \le m)$, $\overline{e_A e_B} = \overline{e_B}\ \overline{e_A}$, $\bar{a} = \sum_A \overline{a_A e_A}$ a conjugation is declared, and $|a| = \left(\sum_A |a_A|^2\right)^{1/2}$.*
2. *$\partial := \sum_{k=0}^{m} e_k \partial/\partial x_k$ and $\bar{\partial} := \sum_{k=0}^{m} \overline{e_k} \partial/\partial x_k$ are the Dirac operator and its conjugate, satisfying $\partial\bar{\partial} = \bar{\partial}\partial = \Delta$.*
3. *If $G \subset \mathbb{R}^{m+1}$ is a regular domain, then a function $f : G \to \mathcal{A}$ is called left-(right-) regular, if $\partial f = 0$ $(f\partial = 0)$ in G.*

Lemma 1 (Rules of differentiation)

1. $$\partial x = x\partial = \bar{\partial}\bar{x} = \bar{x}\bar{\partial} = \sum_{k,\ell=0}^{m} e_k e_\ell \frac{\partial x_k}{\partial x_\ell} = 1 - m,$$
2. $$\partial \bar{x} = \bar{x}\partial = \bar{\partial} x = x\bar{\partial} = \sum_{k,\ell=0}^{m} \overline{e_k} e_\ell \frac{\partial x_k}{\partial x_\ell} = m + 1,$$

3. $\partial|x|^\alpha = |x|^\alpha\partial = \alpha x|x|^{\alpha-2}, \alpha \in \mathbb{R}$,
4. $\bar{\partial}|x|^\alpha = |x|^\alpha\bar{\partial} = \alpha\bar{x}|x|^{\alpha-2}, \alpha \in \mathbb{R}$,
5. $\partial\left(\frac{\bar{x}}{|x|^{m+1}}\right) = \left(\frac{\bar{x}}{|x|^{m+1}}\right)\partial = \bar{\partial}\left(\frac{x}{|x|^{m+1}}\right) = \left(\frac{x}{|x|^{m+1}}\right)\bar{\partial} = 0.$

The proof is straight forward.

The last relation shows that $E(x) := \bar{x}|x|^{-m-1}$ and $x|x|^{-m-1}$ are fundamental solutions to the Dirac Operator and its conjugate leading to Cauchy type kernels.

The following two theorems are well known, so proofs are excluded.

Theorem 1 (STOKES) *If $f, g \in C^1(G, \mathcal{A})$, with G being a differentiable manifold and $\mathrm{d}V = \mathrm{d}x_0 \wedge \ldots \wedge \mathrm{d}x_m$, then*

$$\int\limits_{\partial G} f\mathrm{d}\vec{\sigma}g = \int\limits_{G} \{(f\partial)g + f(\partial g)\}\mathrm{d}V \tag{1}$$

is true.

Theorem 2 (CAUCHY-POMPEIU) *For a regular domain G and an open set G_0 with $\bar{G} \subset G_0$, $f \in \mathbb{C}^1(G_0, \mathcal{A})$ and ω_{m+1} the surface area of the $(m+1)$-dimensional unit ball,*

$$\int\limits_{\partial G} \frac{\overline{(y-x)}}{|y-x|^{m+1}}\mathrm{d}\vec{\sigma}_y f(y) - \int\limits_{G} \frac{\overline{(y-x)}}{|y-x|^{m+1}}(\partial f(y))\mathrm{d}V_y$$

$$= \begin{cases} \omega_{m+1}f(x) & , \text{ if } x \in G \\ 0 & , \text{ if } x \notin \bar{G} \end{cases} \tag{2}$$

and

$$\int\limits_{\partial G} f(y)\ \mathrm{d}\vec{\sigma}_y \frac{\overline{(y-x)}}{|y-x|^{m+1}} - \int\limits_{G} (f(y)\partial)\frac{\overline{(y-x)}}{|y-x|^{m+1}}\mathrm{d}V_y$$

$$= \begin{cases} \omega_{m+1}f(x) & , \text{ if } x \in G \\ 0 & , \text{ if } x \notin \bar{G} \end{cases} . \tag{3}$$

hold.

2 Higher Order Cauchy-Pompeiu representation formulas

In [2] the following higher order Cauchy-Pompeiu formulas are proved. They were sublimited in [3]. Here the formulas from [2] will be generalized. In a forthcoming paper the proofs are given in detail.

In the following let $G \subset \mathbb{R}^{m+1}$ be a regular domain.

Theorem 3 (Iterated representation formula for ∂) *If $k \in \mathbb{N}$ and $f \in C^k(\bar{G}, \mathcal{A})$. Then for $x \in G$*

$$\begin{aligned} f(x) &= \sum_{\mu=0}^{k-1} \frac{(-1)^\mu}{\omega_{m+1}} \int\limits_{\partial G} \frac{(\overline{y-x})(\overline{y-x}+y-x)^\mu}{2^\mu \mu! |y-x|^{m+1}} \mathrm{d}\vec{\sigma}_y \partial^\mu f(y) \\ &\quad + \frac{(-1)^k}{\omega_{m+1}} \int\limits_{G} \frac{(\overline{y-x})(\overline{y-x}+y-x)^{k-1}}{2^{k-1}(k-1)! |y-x|^{m+1}} \partial^k f(y) \mathrm{d}V_y \end{aligned} \tag{4}$$

holds.

Theorem 4 (Iterated representation formula for Δ) *If $1 \leq k$ and $f \in C^{2k}(\bar{G}, \mathcal{A})$, then for $m > 1$, where m is odd with $0 \leq 2k-1 < m$ or m is even,*

$$\begin{aligned} f(x) = \sum_{\mu=1}^{k} \Bigg(& \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{(\overline{y-x})|y-x|^{2(\mu-1)-m-1}}{2^{\mu-1}(\mu-1)! \prod\limits_{j=1}^{\mu-1}(2j-m-1)} \mathrm{d}\vec{\sigma}_y \Big\{ \Delta^{\mu-1} f(y) \Big\} \\ & - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{2\mu-m-1}}{2^{\mu-1}(\mu-1)! \prod\limits_{j=1}^{\mu}(2j-m-1)} \mathrm{d}\overline{\vec{\sigma}}_y \Big\langle \partial \Big\{ \Delta^{\mu-1} f(y) \Big\} \Big\rangle \Bigg) \\ & + \frac{1}{\omega_{m+1}} \int\limits_{G} \frac{|y-x|^{2k-m-1}}{2^{k-1}(k-1)! \prod\limits_{j=1}^{k}(2j-m-1)} \Big\{ \Delta^k f(y) \Big\} \mathrm{d}V_y \end{aligned} \tag{5}$$

holds.

For a general representation formula a general primitive of $\bar{x}|x|^{2k-m-1}$ with respect to ∂ is needed. Because of the non-commutativity the product rule for differentiation is not valid. One has to write general powers of x and $\bar{x}$ as products of powers of x or $\bar{x}$ and $|x|^2$ only, i.e. if $k > \ell$, then rewrite $x^k \bar{x}^\ell$ as $x^{k-\ell}|x|^{2\ell}$. Therefore the proof of the following lemma becomes more involved. In the following let $[x]$ be the entire function of x on $\mathbb{R}$.

Lemma 2 *If $1 \leq k$, then for $\ell \in \mathbb{N}$, $m > 1$ where m is odd with*

$0 \leq 2(k+[(\ell+1)/2])-1 < m$ *or* m *is even,*

$$\frac{\bar{x}^{\ell+1}|x|^{2k-m-1}}{2^{k+\ell}(k+\ell)!\prod\limits_{j=1}^{k}(2j-m-1)}$$

$$+\sum_{n=0}^{[(\ell-1)/2]}\frac{\sum_{\mu=1}^{\ell-2n}\binom{\ell-n-\mu+1}{n+1}\binom{n+\mu-1}{n}\bar{x}^{\ell-\mu-2n}x^{\mu-1}|x|^{2(k+n+1)-m-1}}{2^{k+\ell}(k+\ell)!\prod_{j=n+2}^{k}(2j-m-1)\prod_{j=1}^{n+1}\{2(k+j)-m-1\}}$$

is the ℓ*-th primitive of* $\dfrac{\bar{x}|x|^{2k-m-1}}{2^k k!\prod_{j=1}^{k}(2j-m-1)}$.

The proof is elementary, see [8].

Definition 2 *For even* m *and* $0 \leq 2(k+[(\ell+1)/2])-1 < m, \ell \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ *and* $\nu \in \mathbb{N}$*, let us denote*

$$E_\ell(x) := \frac{\bar{x}^{\ell+1}|x|^{2k-m-1}}{\omega_{m+1}2^{k+\ell}(k+\ell)!\prod_{j=1}^{k}(2j-m-1)}$$

$$+\sum_{n=0}^{[(\ell-1)/2]}\frac{\sum\limits_{\mu=1}^{\ell-2n}\binom{\ell-n-\mu+1}{n+1}\binom{n+\mu-1}{n}\bar{x}^{\ell-\mu-2n}x^{\mu-1}|x|^{2(k+n+1)-m-1}}{\omega_{m+1}2^{k+\ell}(k+\ell)!\prod\limits_{j=n+2}^{k}(2j-m-1)\prod\limits_{j=1}^{n+1}\{2(k+j)-m-1\}},$$

$$E_{-1,\nu-1}(x) := \frac{|x|^{2\nu-m-1}}{\omega_{m+1}2^{\nu-1}(\nu-1)!\prod_{j=1}^{\nu}(2j-m-1)},$$

$$E_{0,\nu}(x) := \frac{\bar{x}|x|^{2\nu-m-1}}{\omega_{m+1}2^{\nu}\nu!\prod_{j=1}^{\nu}(2j-m-1)}$$

and

$$E(x) := E_{0,0}(x).$$

Remark 1 From Lemma 2 follows $\partial E_{\ell+1}(x) = E_\ell(x)$. Moreover, $E_0 = E_{0,k}$ holds and if $\nu \geq 0$

$$\partial E_{0,\nu}(x) \quad = \quad \frac{\partial\left(\dfrac{\bar{x}}{|x|^{m+1}}|x|^{2\nu}\right)}{\omega_{m+1}2^{\nu}\nu!\prod_{j=1}^{\nu}(2j-m-1)}$$

$$= \frac{\frac{\bar{x}}{|x|^{m+1}} 2\nu x|x|^{2(\nu-1)}}{\omega_{m+1}2^{\nu}\nu!\prod_{j=1}^{\nu}(2j-m-1)}$$

$$= \frac{|x|^{2\nu-m-1}}{\omega_{m+1}2^{\nu-1}(\nu-1)!\prod_{j=1}^{\nu}(2j-m-1)} = E_{-1,\nu-1}(x)$$

and

$$\begin{aligned}\bar{\partial}E_{-1,\nu-1}(x) &= \frac{(2\nu-m-1)\bar{x}|x|^{2(\nu-1)-m-1}}{\omega_{m+1}2^{\nu-1}(\nu-1)!\prod_{j=1}^{\nu}(2j-m-1)} \\ &= \frac{|\bar{x}|^{2(\nu-1)-m-1}}{\omega_{m+1}2^{\nu-1}(\nu-1)!\prod_{j=1}^{\nu-1}(2j-m-1)} \\ &= E_{0,\nu-1}(x).\end{aligned}$$

Theorem 5 (General representation formula) *If $f \in C^{2k+\ell+1}(\bar{G};\mathcal{A})$ and $1 \le k$, then for $\ell \in \mathbb{N}_0$, $m > 1$ where m is odd with $0 \le 2(k+[(\ell+1)/2])-1 < m$ or m is even, it holds*

$$\begin{aligned}&f(x)\\ &= \sum_{\nu=0}^{k}\int_{\partial G}\frac{\overline{(y-x)}|y-x|^{2\nu-m-1}}{\omega_{m+1}2^{\nu}\nu!\prod_{j=1}^{\nu}(2j-m-1)}\mathrm{d}\vec{\sigma}_y\{\Delta^{\nu}f(y)\}\\ &+\sum_{\nu=1}^{\ell}(-1)^{\nu}\int_{\partial G}\Bigg\{\frac{\overline{(y-x)}^{\nu+1}|y-x|^{2k-m-1}}{\omega_{m+1}2^{k+\nu}(k+\nu)!\prod_{j=1}^{k}(2j-m-1)}\\ &+\sum_{n=0}^{[(\nu-1)/2]}\frac{\sum_{\mu=1}^{\nu-2n}\binom{\nu-n-\mu+1}{n+1}\binom{n+\mu-1}{n}\overline{(y-x)}^{\nu-\mu-2n}}{\omega_{m+1}2^{k+\nu}(k+\nu)!\prod_{j=n+2}^{k}(2j-m-1)\prod_{j=1}^{n+1}\{2(k+j)-m-1\}}\\ &\times(y-x)^{\mu-1}|y-x|^{2(k+n+1)-m-1}\mathrm{d}\vec{\sigma}_y\left\{\partial^{\nu}\Delta^{k}f(y)\right\}\Bigg\} \qquad (6)\\ &-\sum_{\nu=1}^{k}\int_{\partial G}\frac{|y-x|^{2\nu-m-1}}{\omega_{m+1}2^{\nu-1}(\nu-1)!\prod_{j=1}^{\nu}(2j-m-1)}\mathrm{d}\overline{\vec{\sigma}}_y\{\Delta^{\nu-1}\partial f(y)\}\end{aligned}$$

$$+(-1)^{\ell-1}\int\limits_G\Bigg\{\frac{\overline{(y-x)}^{\ell+1}|x|^{2k-m-1}}{\omega_{m+1}2^{k+\ell}(k+\ell)!\prod\limits_{j=1}^{k}(2j-m-1)}$$

$$+\sum_{n=0}^{[(\ell-1)/2]}\frac{\sum_{\mu=1}^{\ell-2n}\binom{\ell-n-\mu+1}{n+1}\binom{n+\mu-1}{n}\overline{(y-x)}^{\ell-\mu-2n}}{\omega_{m+1}2^{k+\ell}(k+\ell)!\prod\limits_{j=n+2}^{k}(2j-m-1)\prod\limits_{j=1}^{n+1}\{2(k+j)-m-1\}}$$

$$\times(y-x)^{\mu-1}|y-x|^{2(k+n+1)-m-1}\Bigg\}\Big\{\partial^{\ell+1}\Delta^k f(y)\Big\}\,\mathrm{d}V_y\ .$$

Proof The proof follows by induction over ℓ.
1. $\ell=0$: Equation (5) is

$$f(x)=\sum_{\mu=0}^{k-1}\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{(\overline{y-x})|y-x|^{2\mu-m-1}}{2^\mu\mu!\prod\limits_{j=1}^{\mu}(2j-m-1)}\mathrm{d}\vec{\sigma}_y\Big\{\Delta^\mu f(y)\Big\}$$

$$-\sum_{\mu=1}^{k}\frac{1}{\omega_{m+1}}\int\limits_{\partial G}\frac{|y-x|^{2\mu-m-1}}{2^{\mu-1}(\mu-1)!\prod\limits_{j=1}^{\mu}(2j-m-1)}\mathrm{d}\overline{\vec{\sigma}}_y\Big\{\partial\Delta^{\mu-1}f(y)\Big\}$$

$$+\frac{1}{\omega_{m+1}}\int\limits_{G}\frac{|y-x|^{2k-m-1}}{2^{k-1}(k-1)!\prod\limits_{j=1}^{k}(2j-m-1)}\Big\{\Delta^k f(y)\Big\}\mathrm{d}V_y.$$

With $f\in C^{2k+1}(G,\mathcal{A})$, (1) and because of

$$\partial\left\{\frac{\bar{x}|x|^{2k-m-1}}{2^k k!\prod_{j=1}^{k}(2j-m-1)}\right\}=\frac{\partial\left(\frac{\bar{x}}{|x|^{m+1}}|x|^{2k}\right)}{2^k k!\prod_{j=1}^{k}(2j-m-1)}$$

$$=\frac{2k|x|^{2(k-1)}\frac{x\bar{x}}{|x|^{m+1}}}{2^k k!\prod_{j=1}^{k}(2j-m-1)}=\frac{|x|^{2k-m-1}}{2^{k-1}(k-1)!\prod_{j=1}^{k}(2j-m-1)}$$

the area integral is

$$\begin{aligned} I_k(x) &:= \frac{1}{\omega_{m+1}} \int\limits_G \frac{|y-x|^{2k-m-1}}{2^{k-1}(k-1)! \prod_{j=1}^{k}(2j-m-1)} \Big\{ \Delta^k f(y) \Big\} \mathrm{d}V_y \\ &= \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{(\overline{y-x})|y-x|^{2k-m-1}}{2^k k! \prod_{j=1}^{k}(2j-m-1)} \mathrm{d}\vec{\sigma}_y \Big\{ \Delta^k f(y) \Big\} \\ &\quad - \frac{1}{\omega_{m+1}} \int\limits_G \frac{(\overline{y-x})|y-x|^{2k-m-1}}{2^k k! \prod_{j=1}^{k}(2j-m-1)} \Big\{ \partial \Delta^k f(y) \Big\} \mathrm{d}V_y. \end{aligned}$$

It follows

$$\begin{aligned} & f(x) \\ &= \sum_{\mu=1}^{k} \left\{ \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{(\overline{y-x})|y-x|^{2(\mu-1)-m-1}}{2^{\mu-1}(\mu-1)! \prod_{j=1}^{\mu-1}(2j-m-1)} \mathrm{d}\vec{\sigma}_y \Big\{ \Delta^{\mu-1} f(y) \Big\} \right. \\ &\quad \left. - \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{2\mu-m-1}}{2^{\mu-1}(\mu-1)! \prod_{j=1}^{\mu}(2j-m-1)} \mathrm{d}\overline{\vec{\sigma}}_y \partial \Big\{ \Delta^{\mu-1} f(y) \Big\} \right\} \\ &\quad + \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{(\overline{y-x})|y-x|^{2k-m-1}}{2^k k! \prod_{j=1}^{k}(2j-m-1)} \mathrm{d}\vec{\sigma}_y \Delta^k f(y) \\ &\quad - \frac{1}{\omega_{m+1}} \int\limits_G \frac{(\overline{y-x})|y-x|^{2k-m-1}}{2^k k! \prod_{j=1}^{k}(2j-m-1)} \Big\{ \partial \Delta^k f(y) \Big\} \mathrm{d}V_y \\ &= \sum_{\mu=0}^{k} \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{(\overline{y-x})|y-x|^{2\mu-m-1}}{2^\mu \mu! \prod_{j=1}^{\mu}(2j-m-1)} \mathrm{d}\vec{\sigma}_y \Big\{ \Delta^\mu f(y) \Big\} \\ &\quad - \sum_{\mu=1}^{k} \frac{1}{\omega_{m+1}} \int\limits_{\partial G} \frac{|y-x|^{2\mu-m-1}}{2^{\mu-1}(\mu-1)! \prod_{j=1}^{\mu}(2j-m-1)} \mathrm{d}\overline{\vec{\sigma}}_y \Big\{ \partial \Delta^{\mu-1} f(y) \Big\} \\ &\quad - \frac{1}{\omega_{m+1}} \int\limits_G \frac{(\overline{y-x})|y-x|^{2k-m-1}}{2^k k! \prod_{j=1}^{k}(2j-m-1)} \Big\{ \partial \Delta^k f(y) \Big\} \mathrm{d}V_y. \end{aligned}$$

With Definition 2, (6) is equivalent to

$$\begin{aligned} f(x) &= \sum_{\nu=0}^{k} \int_{\partial G} E_{0,\nu}(y-x)\mathrm{d}\vec{\sigma}_y\{\Delta^\nu f(y)\} \\ &\quad + \sum_{\nu=1}^{\ell} (-1)^\nu \int_{\partial G} E_\nu(y-x)\mathrm{d}\vec{\sigma}_y\{\partial^\nu \Delta^k f(y)\} \\ &\quad - \sum_{\nu=1}^{k} \int_{\partial G} E_{-1,\nu-1}(y-x)\mathrm{d}\overline{\vec{\sigma}}_y\{\partial\Delta^{\nu-1} f(y)\} \\ &\quad + (-1)^{\ell+1} \int_G E_\ell(y-x)\{\partial^{\ell+1}\Delta^k f(y)\}\mathrm{d}V_y \ . \end{aligned} \tag{7}$$

2. Let (7) be valid for $\ell \in \mathbb{N}_0$. With (2) applied to $\partial^{\ell+1}\Delta^k f(y)$, it follows

$$\begin{aligned} \partial^{\ell+1}\Delta^k f(y) &= \int_{\partial G} E(z-y)\mathrm{d}\vec{\sigma}_z \partial^{\ell+1}\Delta^k f(z) \\ &\quad - \int_G E(z-y)\left\{\partial^{\ell+2}\Delta^k f(z)\right\}\mathrm{d}V_z. \end{aligned} \tag{8}$$

Let us denote

$$\Psi(x,z) := \int_G E_\ell(y-x)E(y-z)\mathrm{d}V_y$$

and

$$\tilde{\Psi}(x,z) := \int_{\partial G} E_{\ell+1}(y-x)\mathrm{d}\vec{\sigma}_y E(y-z).$$

It follows

$$\tilde{\Psi}(x,z)\partial_z = 0.$$

Using (8) in (7), it follows after changing the order of integration

$$f(x) = \sum_{\nu=0}^{k} \int_{\partial G} E_{0,\nu}(y-x)\mathrm{d}\vec{\sigma}_y\{\Delta^\nu f(y)\}$$

$$
\begin{aligned}
&+\sum_{\nu=1}^{\ell}(-1)^{\nu}\int\limits_{\partial G}E_{\nu}(y-x)\mathrm{d}\vec{\sigma}_y\{\partial^{\nu}\Delta^k f(y)\}\\
&-\sum_{\nu=1}^{k}\int\limits_{\partial G}E_{-1,\nu-1}(y-x)\mathrm{d}\overline{\vec{\sigma}}_y\{\partial\Delta^{\nu-1}f(y)\}\\
&+(-1)^{\ell+1}\int\limits_{\partial G}\Psi(x,z)\mathrm{d}\vec{\sigma}_z\left\{\partial^{\ell+1}\Delta^k f(z)\right\}\\
&-(-1)^{\ell+1}\int\limits_{G}\Psi(x,z)\left\{\partial^{\ell+2}\Delta^k f(z)\right\}\mathrm{d}V_z.
\end{aligned}
$$

With (3) it follows

$$
\begin{aligned}
E_{\ell+1}(z-x)&=\tilde{\Psi}(x,z)+\Psi(x,z),\\
\Psi(x,z)&=E_{\ell+1}(z-x)-\tilde{\Psi}(x,z)
\end{aligned}
$$

and

$$
\begin{aligned}
f(x) \;=\;& \sum_{\nu=0}^{k}\int\limits_{\partial G}E_{0,\nu}(y-x)\mathrm{d}\vec{\sigma}_y\{\Delta^{\nu}f(y)\}\\
&+\sum_{\nu=1}^{\ell}(-1)^{\nu}\int\limits_{\partial G}E_{\nu}(y-x)\mathrm{d}\vec{\sigma}_y\{\partial^{\nu}\Delta^k f(y)\}\\
&-\sum_{\nu=1}^{k}\int\limits_{\partial G}E_{-1,\nu-1}(y-x)\mathrm{d}\overline{\vec{\sigma}}_y\{\partial\Delta^{\nu-1}f(y)\}\\
&+(-1)^{\ell+1}\int\limits_{\partial G}E_{\ell+1}(z-x)\mathrm{d}\vec{\sigma}_z\left\{\partial^{\ell+1}\Delta^k f(z)\right\}\\
&-(-1)^{\ell+1}\int\limits_{G}E_{\ell+1}(z-x)\left\{\partial^{\ell+2}\Delta^k f(z)\right\}\mathrm{d}V_z\\
&-(-1)^{\ell+1}\int\limits_{\partial G}\tilde{\Psi}(x,z)\mathrm{d}\vec{\sigma}_z\left\{\partial^{\ell+1}\Delta^k f(z)\right\}\\
&+(-1)^{\ell+1}\int\limits_{G}\tilde{\Psi}(x,z)\left\{\partial^{\ell+2}\Delta^k f(z)\right\}\mathrm{d}V_z.
\end{aligned}
$$

With (1) it follows

$$\begin{aligned}
& -(-1)^{\ell+1}\int\limits_{\partial G}\tilde{\Psi}(x,z)\mathrm{d}\vec{\sigma}_z\left\{\partial^{\ell+1}\Delta^k f(z)\right\}\\
& +(-1)^{\ell+1}\int\limits_{G}\tilde{\Psi}(x,z)\left\{\partial^{\ell+2}\Delta^k f(z)\right\}\mathrm{d}V_z\\
= \;& (-1)^{\ell}\int\limits_{G}\Big\langle(\tilde{\Psi}(x,z)\partial_z)\{\partial^{\ell+1}\Delta^k f(z)\}+\tilde{\Psi}(x,z)\{\partial^{\ell+2}\Delta^k f(z)\}\Big\rangle\mathrm{d}V_z\\
& -(-1)^{\ell}\int\limits_{G}\tilde{\Psi}(x,z)\{\partial^{\ell+2}\Delta^k f(z)\}\mathrm{d}V_z\\
= \;& 0.
\end{aligned}$$

Thus (6) is valid.

Theorem 6 *Under the conditions of Theorem 5 the weak singular kernel function* $K_{\ell,k}(x) := (-1)^{\ell+1}E_\ell(x)$ *is a fundamental solution in* $\mathcal{A}$ *for* $\partial^{\ell+1}\Delta^k$ *in* $\mathbb{R}^{m+1}$.

This follows from the properties

$$\begin{aligned}
\partial K_{\ell,k}(x) &= -K_{\ell-1,k}(x), \partial K_{0,k}(x)\\
&= -E_{-1,k-1}(x), \Delta E_{-1,k-1}(x) = E_{-1,k-2}\ ,
\end{aligned}$$

and

$$E_{-1,0}(x) = \frac{|x|^{1-m}}{\omega_{m+1}(1-m)}.$$

Theorem 7 *Under the conditions of Theorem* (5)*, for* $f \in L_1(G,\mathcal{A})$

$$T_{\ell,k}f(x) := \int\limits_{G} K_{\ell,k}(y-x)f(y)\mathrm{d}V_y$$

provides a particular solution to $\partial^{\ell+1}\Delta^k w = f$ *in* G *satisfying* $\partial^{\ell+1}\Delta^k w = 0$ *in* $\mathbb{R}^{m+1}\setminus\overline{G}$.

Proof From

$$\partial^{\ell+1}\Delta^{k-1}T_{\ell,k}f(x) = \frac{(-1)^{\ell+1}}{\omega_{m+1}}\int\limits_{G}\frac{|y-x|^{1-m}}{1-m}f(y)\mathrm{d}V_y,$$

follows

$$\partial^{\ell}\Delta^{k}T_{\ell,k}f(x) = (-1)^{\ell}\int\limits_{G}E(y-x)f(y)\mathrm{d}V_y$$

and, see [8],

$$\partial^{\ell+1}\Delta^k T_{\ell,k} f = f$$

in G.

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A revised higher order Cauchy-Pompeiu formulas in Clifford analysis and its application*

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Abstract

A revised higher order Cauchy-Pompeiu formula in Clifford analysis is obtained. In [6], the higher order Cauchy-Pompeiu formula holds only for $k < n$, the reason lies in the kernel functions. In this paper, by constructing suitable kernel functions, the missing integral representations are presented.

AMS Subject Classification: 35C10, 31B05, 30G35

Keywords: Universal Clifford algebra, Cauchy-Pompeiu formula, kernel functions

1 Introduction

The theory of functions with values in a Clifford algebra was well studied by many authors (see [1]-[11], [13], [15], [16] etc.). In 1977, R. Delanghe and F. Brackx firstly introduced the concept of a k-regular function with values in a Clifford algebra $C(V_{n,0})$, the corresponding Cauchy integral formula and Taylor expansions were obtained (see [11]). H. Begehr also obtained different integral representation formulas in Clifford analysis (see [1]-[3] etc.). But the above results hold only for functions with values in $C(V_{n,0})$. Naturally the question arises, how these results can be generalized to functions with values in $C(V_{n,s}), 0 < s \leq n$?

In [5], [12], [17] the Cauchy integral formula and the Cauchy-Pompeiu formula on certain distinguished boundary for functions with values in a universal Clifford algebra $C(V_{n,s})$ and some of its applications were obtained. What can be said about the higher order Cauchy integral formula and Cauchy-Pompeiu formula for functions with values in $C(V_{n,s}), 0 < s \leq n$?

Let D be a bounded domain with smooth boundary ∂D in the complex plane $\mathbb{C}$, and $\omega \in C^1(D,\mathbb{C}) \bigcap C(\overline{D},\mathbb{C})$. The following generalized form of the Cauchy integral formula for the functions of one complex variable is known as the Cauchy-Pompeiu formula [14].

$$\begin{cases} w(z) = \dfrac{1}{2\pi i}\displaystyle\int_{\partial D} \frac{w(\zeta)}{\zeta - z}\mathrm{d}\zeta - \frac{1}{\pi}\iint_D \frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z}\mathrm{d}\xi\mathrm{d}\eta, \\ w(z) = -\dfrac{1}{2\pi i}\displaystyle\int_{\partial D} \frac{w(\zeta)}{\overline{\zeta} - \overline{z}}\mathrm{d}\overline{\zeta} - \frac{1}{\pi}\iint_D \frac{w_{\zeta}(\zeta)}{\overline{\zeta} - \overline{z}}\mathrm{d}\xi\mathrm{d}\eta, \end{cases} \quad \zeta = \xi + i\eta,\ z \in D,$$

where the Wirtinger-Kolossov operators $\dfrac{\partial}{\partial\zeta}, \dfrac{\partial}{\partial\overline{\zeta}}$ are given via

$$w_{\zeta} = \frac{\partial w}{\partial\zeta} = \frac{1}{2}\left(\frac{\partial w}{\partial\xi} - i\frac{\partial w}{\partial\eta}\right), \quad w_{\overline{\zeta}} = \frac{\partial w}{\partial\overline{\zeta}} = \frac{1}{2}\left(\frac{\partial w}{\partial\xi} + i\frac{\partial w}{\partial\eta}\right).$$

In [6], we gave the higher order Cauchy-Pompeiu formula for functions with values in $C(V_{n,n})$, but this holds only for $k < n$. What can be said if $k \geq n$?

In this paper, the revised higher order Cauchy-Pompeiu formula and the higher order Cauchy integral formula are presented for functions with values in the Clifford algebra $C(V_{n,n})$.

2 Preliminaries and notations

Let $V_{n,n}$ be an n–dimensional ($n \geq 1$) real linear space with basis $\{e_1, e_2, \cdots, e_n\}$, $C(V_{n,n})$ be the 2^n–dimensional real linear space with basis

$$\{e_A, A = \{h_1, \cdots, h_r\} \in \mathcal{P}N, 1 \leq h_1 < \cdots < h_r \leq n\},$$

where N stands for the set $\{1, \cdots, n\}$ and $\mathcal{P}N$ denotes for the family of all order-preserved subsets of N in the above way. We denote $e_\emptyset$ as e_0 and e_A as $e_{h_1 \cdots h_r}$ for $A = \{h_1, \cdots, h_r\} \in \mathcal{P}N$. The product on $C(V_{n,n})$ is defined by

$$\begin{cases} e_A e_B = (-1)^{\#((A\cap B)\setminus N)}(-1)^{P(A,B)} e_{A\triangle B}, \ \text{ if } \ A, B \in \mathcal{P}N, \\ \\ \lambda\mu = \sum\limits_{A\in\mathcal{P}N} \sum\limits_{B\in\mathcal{P}N} \lambda_A \mu_B e_A e_B, \quad \text{if } \ \lambda = \sum\limits_{A\in\mathcal{P}N} \lambda_A e_A, \ \mu = \sum\limits_{B\in\mathcal{P}N} \mu_B e_B. \end{cases} \tag{1}$$

where $\#(A)$ is the cardinal number of the set A, the number $P(A, B) = \sum\limits_{j\in B} P(A, j)$, $P(A, j) = \#\{i, i \in A, i > j\}$, the symmetric difference set $A\triangle B$ is also order-preserved in the above way, and $\lambda_A \in \mathbb{R}$ is the coefficient of the e_A–component of the Clifford number λ. It follows at once from the multiplication rule (1) that e_0 is the identity element written now as 1 and in particular,

$$\begin{cases} e_i^2 = 1, & \text{if } \ i = 1, \cdots, n, \\ e_i e_j = -e_j e_i, & \text{if } \ 1 \leq i < j \leq n, \\ e_{h_1} e_{h_2} \cdots e_{h_r} = e_{h_1 h_2 \cdots h_r}, & \text{if } \ 1 \leq h_1 < h_2 \cdots, < h_r \leq n. \end{cases} \tag{2}$$

Thus $C(V_{n,n})$ is a real linear, associative, but non-commutative algebra and it is called the universal Clifford algebra over $V_{n,n}$.

Frequent use will be made of the notation $\mathbb{R}^n_z = \mathbb{R}^n \setminus \{z\}$ where $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$, which means to remove z from $\mathbb{R}^n$. In particular $\mathbb{R}^n_0 = \mathbb{R}^n \setminus \{(0, \cdots, 0)\}$.

To avoid the discussion for the trival case, we always assume $n \geq 2$. Let Ω be an open non empty subset of $\mathbb{R}^n$. We shall consider the operator D which is written as follows,

$$D = \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k} : \ C^{(r)}(\Omega, C(V_{n,n})) \to C^{(r-1)}(\Omega, C(V_{n,n})).$$

Let f be a function with values in $C(V_{n,n})$ defined in Ω, the operator D acts on functions f from the left and right being governed by the rule

$$D[f]=\sum_{k=1}^{n}\sum_{A}e_ke_A\frac{\partial f_A}{\partial x_k},\quad [f]D=\sum_{k=1}^{n}\sum_{A}e_Ae_k\frac{\partial f_A}{\partial x_k}.$$

Definition 1 *A function $f\in C^{(r)}(\Omega,C\left(V_{n,n}\right))$ $(r\geq 1)$ is called left (right) regular in Ω if $D[f]=0\,([f]D=0)$ in Ω; A function $f\in C^{(r)}(\Omega,C\left(V_{n,n}\right))$ $(r\geq k)$ is called left (right) k-regular in Ω if $D^k[f]=0\,([f]D^k=0)$ in Ω.*

In the sequel, we shall only consider the left k-regular functions, we shall denote them as k-regular function for brevity; the right k-regular functions can be similarly discussed. We shall also use the following $C\left(V_{n.n}\right)$–valued $(n-1)$–differential forms, which are exact and written as

$$\mathrm{d}\sigma=\sum_{k=1}^{n}(-1)^{k-1}e_k\mathrm{d}\widehat{x}_k^N,$$

where

$$\mathrm{d}\widehat{x}_k^N=\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^{k-1}\wedge\mathrm{d}x^{k+1}\cdots\wedge\mathrm{d}x^n.$$

3 Integral representations

In [6], we have constructed the kernel functions for $1\leq j<n$ as follows,

$$H_j^*(x)=\frac{A_j}{\omega_n}\frac{\mathbf{x}^j}{\rho^n(x)},\qquad 1\leq j<n,\ \ x\in\mathbb{R}_0^n, \tag{3}$$

where $\mathbf{x}=\sum\limits_{k=1}^{n}x_ke_k$, $\rho(x)=\left(\sum\limits_{k=1}^{n}x_k^2\right)^{\frac{1}{2}}$, and ω_n denotes the area of the unit sphere in $\mathbb{R}^n$, and

$$A_j=\frac{1}{2^{[\frac{j-1}{2}]}[\frac{j-1}{2}]!\prod\limits_{r=1}^{[\frac{j}{2}]}(2r-n)},\qquad 1\leq j<n. \tag{4}$$

Let $H_j^*(x)$ be as above. In [6], it is proved that

$$\begin{cases} D\left[H_1^*(x)\right] & =\left[H_1^*(x)\right]D=0,\ \ x\in\mathbb{R}_0^n,\\ D\left[H_{j+1}^*(x)\right] & =\left[H_{j+1}^*(x)\right]D=H_j^*(x),\ x\in\mathbb{R}_0^n,\\ & \qquad\qquad \textit{for any } 1\leq j<n-1. \end{cases} \tag{5}$$

Now, for $x \in \mathbb{R}_0^n$, we define $H_j^*(x)$ for any $j \geq 1$ as follows,

$$H_j^*(x) = \begin{cases} \dfrac{A_j}{\omega_n}\dfrac{\mathbf{x}^j}{\rho^n(x)}, & n \text{ is odd;} \\ \dfrac{A_j}{\omega_n}\dfrac{\mathbf{x}^j}{\rho^n(x)}, & 1 \leq j < n,\ n \text{ is even;} \\ \dfrac{A_{j-1}}{2\omega_n}\log(\mathbf{x}^2), & j = n,\ n \text{ is even;} \\ \dfrac{A_{n-1}}{2\omega_n}C_{l,0}\mathbf{x}^l\left(\log(\mathbf{x}^2) - 2\displaystyle\sum_{i=0}^{l-1}\frac{C_{i+1,0}}{C_{i,0}}\right), & j = n+l, l > 0, \\ & n \text{ is even;} \end{cases} \tag{6}$$

where for any $j \geq 1$, when n is odd or for $1 \leq j < n$, when n is even, A_j is defined as (4) and

$$C_{j,0} = \begin{cases} 1, & j = 0, \\ \dfrac{1}{2^{[\frac{j}{2}]}\left([\frac{j}{2}]\right)!\displaystyle\prod_{\mu=0}^{[\frac{j-1}{2}]}(n+2\mu)}, & j \in \mathbf{N}^* = \mathbf{N}\backslash\{0\}. \end{cases} \tag{7}$$

Theorem 1 *Let $H_j^*(x)$ be as (6) for any $j \geq 1$, then*

$$\begin{cases} D\left[H_1^*(x)\right] & = \left[H_1^*(x)\right] D = 0,\ x \in \mathbb{R}_0^n, \\ D\left[H_{j+1}^*(x)\right] & = \left[H_{j+1}^*(x)\right] D = H_j^*(x),\ x \in \mathbb{R}_0^n,\ \textit{for any}\ j \geq 1. \end{cases} \tag{8}$$

Proof Firstly, in the case of n being an odd number or being an even number with $j < n-1$, it may be similarly proved as in [6], and so it is enough to consider the case of n being an even number with $j \geq n-1$, in the following, we assume n to be an even number, $x \in \mathbb{R}_0^n$.

Secondly, we shall show that,

$$D\left[H_n^*(x)\right] = \left[H_n^*(x)\right] D = H_{n-1}^*(x),\ x \in \mathbb{R}_0^n. \tag{9}$$

It is easy to check that,

$$D\left[\log(\mathbf{x}^2)\right] = \left[\log(\mathbf{x}^2)\right] D = \frac{2\mathbf{x}}{\rho^2}, \tag{10}$$

thus (9) holds.We continue to show that, for $l > 0$,

$$D\left[C_{l,0}\mathbf{x}^l \log(\mathbf{x}^2)\right] = \left[C_{l,0}\mathbf{x}^l \log(\mathbf{x}^2)\right]$$

$$D = C_{l-1,0}\mathbf{x}^{l-1}\log(\mathbf{x}^2) + 2C_{l,0}\mathbf{x}^{l-1}, \tag{11}$$

It may be deduced directly that, for $l > 0$,

$$D\left[C_{l,0}\mathbf{x}^l\right] = \left[C_{l,0}\mathbf{x}^l\right] D = C_{l-1,0}\mathbf{x}^{l-1}. \tag{12}$$

In the following, we shall prove (11) by induction.

Step 1. For $l = 1$, by (10) and (12), we have,

$$\begin{aligned} & D\left[C_{1,0}\mathbf{x}\log(\mathbf{x}^2)\right] \\ = \ & \log(\mathbf{x}^2) + C_{1,0}\sum_{i=1}^{n} e_i\mathbf{x}\frac{\partial}{\partial x_i}\left(\log(\mathbf{x}^2)\right) \\ = \ & \log(\mathbf{x}^2) + C_{1,0}\sum_{i=1}^{n}\left(-\mathbf{x}e_i + 2x_i\right)\frac{\partial}{\partial x_i}\left(\log(\mathbf{x}^2)\right) \\ = \ & \log(\mathbf{x}^2) + 2C_{1,0}. \end{aligned}$$

It may be similarly proved that,

$$\left[C_{1,0}\mathbf{x}\log(\mathbf{x}^2)\right] D = \log(\mathbf{x}^2) + 2C_{1,0}.$$

Step 2. Suppose (11) holds for $l = k$, in view of $\log(\mathbf{x}^2)$ being a scalar function, by (10) and (12) again, we have,

$$\begin{aligned} & D\left[C_{k+1,0}\mathbf{x}^{k+1}\log(\mathbf{x}^2)\right] = \left[C_{k+1,0}\mathbf{x}^{k+1}\log(\mathbf{x}^2)\right] D \\ & = D\left[\log(\mathbf{x}^2)\right] C_{k+1,0}\mathbf{x}^{k+1} + D\left[C_{k+1,0}\mathbf{x}^{k+1}\right]\log(\mathbf{x}^2) \\ & = \frac{2\mathbf{x}}{\rho^2}C_{k+1,0}\mathbf{x}^{k+1} + C_{k,0}\mathbf{x}^k\log(\mathbf{x}^2) \\ & = C_{k,0}\mathbf{x}^k\log(\mathbf{x}^2) + 2C_{k+1,0}\mathbf{x}^k, \end{aligned}$$

thus, (11) holds.

At last, we shall show that, for $l > 0$,

$$D\left[H^*_{n+l}(x)\right] = \left[H^*_{n+l}(x)\right] D = H^*_{n+l-1}(x),\ x \in \mathbb{R}^n_0. \tag{13}$$

For $l = 1$, by (11) and (12), we have,

$$\begin{aligned} D\left[H^*_{n+1}(x)\right] & = \left[H^*_{n+1}(x)\right] D \\ & = \frac{A_{n-1}}{2\omega_n}\left(\log(\mathbf{x}^2) + 2C_{1,0} - 2C_{0,0}\cdot\frac{C_{1,0}}{C_{0,0}}\right) \\ & = H^*_n(x). \end{aligned}$$

Thus (13) holds for $l = 1$. We continue to show that (13) holds for $l > 1$. By (11) and (12) again, we have,

$$\begin{aligned}
&D\left[H_{n+l}^*(x)\right] = \left[H_{n+l}^*(x)\right] D \\
&= \frac{A_{n-1}}{2\omega_n}\left(C_{l-1,0}\mathbf{x}^{l-1}\log(\mathbf{x}^2) + 2C_{l,0}\mathbf{x}^{l-1} - 2C_{l-1,0}\mathbf{x}^{l-1}\sum_{i=0}^{l-1}\frac{C_{i+1,0}}{C_{i,0}}\right) \\
&= \frac{A_{n-1}}{2\omega_n}C_{l-1,0}\mathbf{x}^{l-1}\left(\log(\mathbf{x}^2) - 2\sum_{i=0}^{l-2}\frac{C_{i+1,0}}{C_{i,0}}\right) \\
&= H_{n+l-1}^*(x).
\end{aligned}$$

Thus the result follows.

Corollary 1 *Let $H_k^*(x)(k \geq 1)$ be as above, then we have,*

$$\begin{cases} D^k\left[H_k^*(x)\right] &= \left[H_k^*(x)\right] D^k = 0,\ x \in \mathbb{R}_0^n, \\ D^j\left[H_k^*(x)\right] &= \left[H_k^*(x)\right] D^j = H_{k-j}^*(x),\ x \in \mathbb{R}_0^n,\ 1 \leq j < k. \end{cases} \tag{14}$$

Proof It may be directly proved by Theorem 1.

Corollary 2 *Let $H_k^*(x)(k \geq 1)$ be as above, then we have,*

$$\begin{cases} D^k\left[H_k^*(x-z)\right] &= \left[H_k^*(x-z)\right] D^k = 0,\ x \in \mathbb{R}_z^n, \\ \\ D^j\left[H_k^*(x-z)\right] &= \left[H_k^*(x-z)\right] D^j = H_{k-j}^*(x-z),\ x \in \mathbb{R}_z^n, \\ & \qquad\qquad\qquad\qquad\qquad 1 \leq j < k. \end{cases} \tag{15}$$

Proof It may be directly proved by Theorem 3.1.

Remark 1 From Corollary 1 and Corollary 2, $H_k^*(x)(k \geq 1)$ are left (right) k-regular functions with values in the universal Clifford algebra $C(V_{n,n})$ in $\mathbb{R}_0^n$; $H_k^*(x-z)(k \geq 1)$ are left (right) k-regular functions with values in the universal Clifford algebra $C(V_{n,n})$ in $\mathbb{R}_z^n$.

Theorem 2 (Higher order Cauchy-Pompeiu formula) *Suppose that M is an n–dimensional differentiable compact oriented manifold contained in some open non empty subset $\Omega \subset \mathbb{R}^n$, $f \in C^{(r)}(\Omega, C$*

$(V_{n,n}))$, $r \geq k$, $k \geq 1$. *Moreover,* ∂M *is given the induced orientation. For each* $j = 1, \cdots, k$, $H_j^*(x)$ *is as above. Then, for* $z \in \overset{\circ}{M}$

$$f(z) = \sum_{j=0}^{k-1}(-1)^j \int_{\partial M} H_{j+1}^*(x-z)\mathrm{d}\sigma D^j f(x) + (-1)^k \int_M H_k^*(x-z) D^k f(x)\mathrm{d}x^N. \tag{16}$$

Proof In view of the weak singularity and the properties of (8) of the kernel $H_j^*(x)$, $j \geq 1$, by Stokes formula, (16) may be similarly proved as in [6].

In the following, we shall give an application of the revised higher order Cauchy-Pompeiu formula, namely the higher order Cauchy integral formula.

Theorem 3 (Cauchy integral formula) *Suppose that* M *is an* n*–dimensional differentiable compact oriented manifold contained in some open non empty subset* $\Omega \subset \mathbb{R}^n$*, and let* f *be a left* k*-regular function in* Ω*,* $k \geq 1$*. Moreover,* ∂M *is given the induced orientation, and for each* $j = 1, \cdots, k$, $H_j^*(x)$ *is as above. Then*

$$\sum_{j=0}^{k-1}(-1)^j \int_{\partial M} H_{j+1}^*(x-z)\mathrm{d}\sigma D^j f(x) = \begin{cases} 0, & \text{if } z \notin M, \\ f(z), & \text{if } z \in \overset{\circ}{M}. \end{cases} \tag{17}$$

Proof By Theorem 2 and Stokes formula, in view of the function f being a left k-regular function in M, the result follows.

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A general higher-order integral representation formula for polydomains

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Abstract

Integral representation formulas are known in almost every case for solutions of second–order systems in polydomains composed by as well the Laplace as the Bitsadze operator in a single variable (see [2], [3], [4]). Additionally, representations of solutions to some special kinds of mixed systems of first and second order are available. The above mentioned results are achieved by an extended use of the complex version of the Gauss theorem. A general higher–order representation formula in polydomains can be given by the same method.

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1 Prerequisites

Let $\mathrm{G} \subset \mathbb{C}$ be a bounded domain with smooth boundary. As a consequence of the complex version of the celebrated Gauss theorem (see [1], [7])

$$\frac{1}{2\pi}\int\limits_{\partial G}\omega(z)d((iz)^{\tau_1}(-i\overline{z})^{\tau_2}) = -\frac{1}{\pi}\int\limits_{G}\partial_{\overline{z}}^{\tau_1}\partial_{z}^{\tau_2}\omega(z)dxdy,$$

where $(\tau_1,\tau_2)\in\{(1,0),(0,1)\}$ and $\omega\in \mathrm{C}^1(\mathrm{G};\mathbb{C})\cap\mathrm{C}(\overline{\mathrm{G}};\mathbb{C})$, the Cauchy–Pompeiu representations

$$\begin{aligned}
\omega(z) &= \frac{1}{2\pi i}\int\limits_{\partial \mathrm{G}}\omega(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{\mathrm{G}}\omega_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z},\\
\omega(z) &= -\frac{1}{2\pi i}\int\limits_{\partial \mathrm{G}}\omega(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int\limits_{\mathrm{G}}\omega_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}}
\end{aligned}$$

for $z\in \mathrm{G}$ result (see [1]). An iteration of these formulas leads to a higher–order representation in terms of integral operators of a certain hierarchy (see [5]). To formulate this representation accurately, the following definitions are useful:

Definition 1 *For $m,n\in\mathbb{Z}$ with $m+n\geq 0$ and $(m,n)\neq(0,0)$ let*

$$K_{m,n}(z) := \begin{cases}
\dfrac{(-1)^m(-m)!}{\pi(n-1)!}z^{m-1}\overline{z}^{n-1}, & m\leq 0,\\[2ex]
\dfrac{(-1)^n(-n)!}{\pi(m-1)!}z^{m-1}\overline{z}^{n-1}, & n\leq 0,\\[2ex]
\dfrac{z^{m-1}\overline{z}^{n-1}}{\pi(m-1)!(n-1)!}\left[\log|z|^2 - \displaystyle\sum_{\mu=1}^{m-1}\frac{1}{\mu} - \sum_{\nu=1}^{n-1}\frac{1}{\nu}\right], & 1\leq m,n.
\end{cases}$$

Definition 2 *For $\omega\in \mathrm{L}_1(\mathrm{G})$ and $m,n\in\mathbb{Z}$ with $m+n\geq 0$ let*

$$\mathrm{T}_{\mathrm{G},0,0}\omega(z) := \omega(z) \quad \textit{for} \quad (m,n)=(0,0),$$

$$\mathrm{T}_{\mathrm{G},m,n}\omega(z) := \int\limits_{G}K_{m,n}(z-\zeta)\omega(\zeta)d\xi d\eta \quad \textit{for} \quad (m,n)\neq(0,0).$$

Now everything is prepared to state

Theorem 1 *Let* $\omega \in \mathrm{C}^n(\mathrm{G};\mathbb{C}) \cap \mathrm{C}^{n-1}(\overline{\mathrm{G}};\mathbb{C})$ *and*

$$(\mu_m, \nu_m), \quad 0 \le m \le k,$$

be a chain of bi–indices with $\mu_0 = \nu_0 = 0$, $\mu_{m-1} \le \mu_m$, $\nu_{m-1} \le \nu_m$, $\mu_{m-1} + \nu_{m-1} + 1 = \mu_m + \nu_m$, $1 \le m \le k$. *Then for* $z \in \mathrm{G}$

$$\begin{aligned} \omega(z) &= \mathrm{T}_{\mathrm{G},\mu_k,\nu_k} \partial_z^{\mu_k} \partial_{\overline{z}}^{\nu_k} \omega(z) \\ &\quad + \sum_{m=0}^{k-1} \frac{1}{2} \int\limits_{\partial \mathrm{G}} K_{\mu_{m+1},\nu_{m+1}}(z-\zeta) \partial_\zeta^{\mu_m} \partial_{\overline{\zeta}}^{\nu_m} \omega(\zeta) \\ &\quad \times d\left[(i\zeta)^{\nu_{m+1}-\nu_m} (-i\overline{\zeta})^{\mu_{m+1}-\mu_m} \right]. \end{aligned}$$

2 The general representation formula

The statement of Theorem 1 successively applied to each single variable of G^n gives (see [6])

Theorem 2 *For* $n \in \mathbb{N}$ *and* $k_j \in \mathbb{N}$, $1 \le j \le n$, *let*

$$(\mu_{m_j}^j, \nu_{m_j}^j), \quad 0 \le m_j \le k_j,$$

be chains of bi–indices with

$$\begin{aligned} &(\mu_0^j, \nu_0^j) = (0,0), \\ &\mu_{m_j-1}^j \le \mu_{m_j}^j, \nu_{m_j-1}^j \le \nu_{m_j}^j, \\ &\mu_{m_j-1}^j + \nu_{m_j-1}^j + 1 = \mu_{m_j}^j + \nu_{m_j}^j, 1 \le m_j \le k_j, \end{aligned} \tag{1}$$

and $\mathrm{G}_j \subset \mathbb{C}$ *be bounded domains with smooth boundaries. Then for every function*

$$\omega : \partial_0 \mathrm{G}^n \cup \mathrm{G}^n \to \mathbb{C}, \quad \mathrm{G}^n := \mathrm{G}_1 \times \ldots \times \mathrm{G}_n,$$

(where $\partial_0 \mathrm{G}^n := \partial \mathrm{G}_1 \times \ldots \times \partial \mathrm{G}_n$ *denotes the distinguished boundary of* G^n*) with the differentiability conditions*

$$\begin{aligned} \partial_{z_1}^{\mu_{m_1}^1} \partial_{\overline{z}_1}^{\nu_{m_1}^1} \omega &= f_{m_1} \in \mathrm{L}_1(\mathrm{G}^n;\mathbb{C}), \quad 0 \le m_1 \le k_1, \ldots, \\ \prod_{\rho=1}^{n} \partial_{z_\rho}^{\mu_{m_\rho}^\rho} \partial_{\overline{z}_\rho}^{\nu_{m_\rho}^\rho} \omega &= f_{m_1,\ldots,m_n} \in \mathrm{L}_1(\mathrm{G}^n;\mathbb{C}), \quad 0 \le m_j \le k_j, \\ &\qquad\qquad 1 \le j \le n, \quad 2 \le n, \end{aligned}$$

in G,

$$\begin{aligned}\partial_{z_1}^{\mu_{k_1-1}^1}\partial_{\overline{z}_1}^{\nu_{k_1-1}^1}\omega &= \varphi_{k_1-1}\in \mathrm{C}(\partial_0\mathrm{G}^n;\mathbb{C}),\dots,\\ \prod_{\rho=1}^{n}\partial_{z_\rho}^{\mu_{k_\rho-1}^\rho}\partial_{\overline{z}_\rho}^{\nu_{k_\rho-1}^\rho}\omega &= \varphi_{k_1-1,\dots,k_{n-1}-1}\in \mathrm{C}(\partial_0\mathrm{G}^n;\mathbb{C}),\end{aligned}$$

on $\partial_0\mathrm{G}^n$*, with*

$$\begin{aligned}\partial_{z_2}^{\mu_{k_2}^2}\partial_{\overline{z}_2}^{\nu_{k_2}^2}f_{m_1} &= f_{m_1,k_2},\quad 0\le m_1\le k_1-1,\dots,\\ \partial_{z_n}^{\mu_{k_n}^n}\partial_{\overline{z}_n}^{\nu_{k_n}^n}f_{m_1,\dots,m_{n-1}} &= f_{m_1,\dots,m_{n-1},k_n},\ 0\le m_j\le k_j-1,\\ &\qquad\qquad 1\le j\le n-1,\end{aligned}$$

it holds

$$\begin{aligned}\omega(z) = &\sum_{\substack{(p_1,\dots,p_n)\in\{0,1\}^n\\(p_1,\dots,p_n)\neq(0,\dots,0)}}(-1)^{1+\sum_{j=1}^n p_j}\prod_{j=1}^{n}\left(\mathrm{T}_{\mathrm{G}_j,\mu_{k_j}^j,\nu_{k_j}^j}\right)^{p_j}\\ &\times\prod_{j=1}^{n}\left(\partial_{z_j}^{\mu_{k_j}^j}\partial_{\overline{z}_j}^{\nu_{k_j}^j}\right)^{p_j}\omega(z)\\ &+\frac{1}{2^n}\prod_{j=1}^{n}\sum_{m_j=0}^{k_j-1}\int_{\partial_0\mathrm{G}^n}\prod_{j=1}^{n}\mathrm{K}_{\mu_{m_j+1}^j,\nu_{m_j+1}^j}(z_j-\zeta_j)\\ &\times\prod_{j=1}^{n}\partial_{\zeta_j}^{\mu_{m_j}^j}\partial_{\overline{\zeta}_j}^{\nu_{m_j}^j}\omega(\zeta)\\ &\times\prod_{j=1}^{n}d\left[(i\zeta_j)^{\nu_{m_j+1}^j-\nu_{m_j}^j}(-i\overline{\zeta}_j)^{\mu_{m_j+1}^j-\mu_{m_j}^j}\right],\end{aligned}\tag{2}$$

where the abbreviation

$$\prod_{j=1}^{n}\sum_{m_j=0}^{k_j-1}:=\sum_{m_1=0}^{k_1-1}\cdots\sum_{m_n=0}^{k_n-1}$$

is used.

Proof For $n = 1$ the statement of the theorem coincides with the formula in Theorem 1. For an additional chain of bi–indices

$$(\mu_{m_{n+1}}^{n+1}, \nu_{m_{n+1}}^{n+1}), \quad 0 \le m_{n+1} \le k_{n+1},$$

with the properties (1) let $\omega : \partial_0 \mathrm{G}^{n+1} \cup \mathrm{G}^{n+1} \to \mathbb{C}$ be a function with the differentiability conditions

$$\begin{aligned} \partial_{z_1}^{\mu_{m_1}^1} \partial_{\overline{z}_1}^{\nu_{m_1}^1} \omega &= f_{m_1} \in \mathrm{L}_1(\mathrm{G}^{n+1}; \mathbb{C}), \quad 0 \le m_1 \le k_1, \dots, \\ \prod_{\rho=1}^{n+1} \partial_{z_\rho}^{\mu_{m_\rho}^\rho} \partial_{\overline{z}_\rho}^{\nu_{m_\rho}^\rho} \omega &= f_{m_1,\dots,m_{n+1}} \in \mathrm{L}_1(\mathrm{G}^{n+1}; \mathbb{C}), \quad 0 \le m_j \le k_j, \\ & \qquad\qquad 1 \le j \le n+1, \quad 1 \le n, \end{aligned}$$

in G^{n+1},

$$\begin{aligned} \partial_{z_1}^{\mu_{k_1-1}^1} \partial_{\overline{z}_1}^{\nu_{k_1-1}^1} \omega &= \varphi_{k_1-1} \in \mathrm{C}(\partial_0 \mathrm{G}^{n+1}; \mathbb{C}), \dots, \\ \prod_{\rho=1}^{n+1} \partial_{z_\rho}^{\mu_{k_\rho-1}^\rho} \partial_{\overline{z}_\rho}^{\nu_{k_\rho-1}^\rho} \omega &= \varphi_{k_1-1,\dots,k_n-1} \in \mathrm{C}(\partial_0 \mathrm{G}^{n+1}; \mathbb{C}), \end{aligned}$$

on $\partial_0 \mathrm{G}^{n+1}$, with

$$\begin{aligned} \partial_{z_2}^{\mu_{k_2}^2} \partial_{\overline{z}_2}^{\nu_{k_2}^2} f_{m_1} &= f_{m_1,k_2}, \quad 0 \le m_1 \le k_1 - 1, \dots, \\ \partial_{z_{n+1}}^{\mu_{k_{n+1}}^{n+1}} \partial_{\overline{z}_{n+1}}^{\nu_{k_{n+1}}^{n+1}} f_{m_1,\dots,m_n} &= f_{m_1,\dots,m_n,k_{n+1}}, \quad 0 \le m_j \le k_j - 1, \\ & \qquad\qquad 1 \le j \le n. \end{aligned} \tag{3}$$

Then ω is according to (2) in $\mathrm{G}^n \times \mathrm{G}_{n+1}$ representable as

$$\begin{aligned} &\omega(z_1, \dots, z_n, z_{n+1}) \\ &\quad = \sum_{\substack{(p_1,\dots,p_n) \in \{0,1\}^n \\ (p_1,\dots,p_n) \ne (0,\dots,0)}} (-1)^{1+\sum_{j=1}^n p_j} \prod_{j=1}^n \left(\mathrm{T}_{\mathrm{G}_j, \mu_{k_j}^j, \nu_{k_j}^j} \right)^{p_j} \\ &\qquad \times \prod_{j=1}^n \left(\partial_{z_j}^{\mu_{k_j}^j} \partial_{\overline{z}_j}^{\nu_{k_j}^j} \right)^{p_j} \omega(z_1, \dots, z_n, z_{n+1}) \\ &\qquad + \frac{1}{2^n} \prod_{j=1}^n \sum_{m_j=0}^{k_j-1} \int_{\partial_0 \mathrm{G}^n} \prod_{j=1}^n \mathrm{K}_{\mu_{m_j+1}^j, \nu_{m_j+1}^j}(z_j - \zeta_j) \end{aligned} \tag{4}$$

$$\times \prod_{j=1}^{n} \partial_{\zeta_j}^{\mu_{m_j}^j} \partial_{\overline{\zeta}_j}^{\nu_{m_j}^j} \omega(\zeta_1, \ldots, \zeta_n, z_{n+1})$$
$$\times \prod_{j=1}^{n} d\left[(i\zeta_j)^{\nu_{m_j+1}^j - \nu_{m_j}^j} (-i\overline{\zeta}_j)^{\mu_{m_j+1}^j - \mu_{m_j}^j} \right].$$

Because of (3) all partial derivatives of

$$\partial_{z_{n+1}}^{\mu_{k_{n+1}}^{n+1}} \partial_{\overline{z}_{n+1}}^{\nu_{k_{n+1}}^{n+1}} \prod_{j=1}^{n} \partial_{\zeta_j}^{\mu_{m_j}^j} \partial_{\overline{\zeta}_j}^{\nu_{m_j}^j} \omega(\zeta_1, \ldots, \zeta_n, z_{n+1})$$

exist in G_{n+1}. Therefore the function

$$\prod_{j=1}^{n} \partial_{\zeta_j}^{\mu_{m_j}^j} \partial_{\overline{\zeta}_j}^{\nu_{m_j}^j} \omega(\zeta_1, \ldots, \zeta_n, z_{n+1})$$

has via Theorem 1 in G_{n+1} the representation

$$\begin{aligned}
&\prod_{j=1}^{n} \partial_{\zeta_j}^{\mu_{m_j}^j} \partial_{\overline{\zeta}_j}^{\nu_{m_j}^j} \omega(\zeta_1, \ldots, \zeta_n, z_{n+1}) \\
&= \mathrm{T}_{\mathrm{G}_{n+1}, \mu_{k_{n+1}}^{n+1}, \nu_{k_{n+1}}^{n+1}} \partial_{z_{n+1}}^{\mu_{k_{n+1}}^{n+1}} \partial_{\overline{z}_{n+1}}^{\nu_{k_{n+1}}^{n+1}} \prod_{j=1}^{n} \partial_{\zeta_j}^{\mu_{m_j}^j} \partial_{\overline{\zeta}_j}^{\nu_{m_j}^j} \omega(\zeta_1, \ldots, \zeta_n, z_{n+1}) \\
&\quad + \sum_{m_{n+1}=0}^{k_{n+1}-1} \frac{1}{2} \int_{\partial \mathrm{G}_{n+1}} \mathrm{K}_{\mu_{m_{n+1}+1}^{n+1}, \nu_{m_{n+1}+1}^{n+1}}(z_{n+1} - \zeta_{n+1}) \\
&\quad \times \partial_{\zeta_{n+1}}^{\mu_{m_{n+1}}^{n+1}} \partial_{\overline{\zeta}_{n+1}}^{\nu_{m_{n+1}}^{n+1}} \prod_{j=1}^{n} \partial_{\zeta_j}^{\mu_{m_j}^j} \partial_{\overline{\zeta}_j}^{\nu_{m_j}^j} \omega(\zeta_1, \ldots, \zeta_n, \zeta_{n+1}) \\
&\quad \times d\left[(i\zeta_{n+1})^{\nu_{m_{n+1}+1}^{n+1} - \nu_{m_{n+1}}^{n+1}} (-i\overline{\zeta}_{n+1})^{\mu_{m_{n+1}+1}^{n+1} - \mu_{m_{n+1}}^{n+1}} \right].
\end{aligned} \tag{5}$$

Inserting (5) in (4) leads to

$$\begin{aligned}
&\omega(z_1, \ldots, z_n, z_{n+1}) \\
&\quad = \sum_{\substack{(p_1, \ldots, p_n) \in \{0,1\}^n \\ (p_1, \ldots, p_n) \neq (0, \ldots, 0)}} (-1)^{1 + \sum_{j=1}^{n} p_j} \prod_{j=1}^{n} \left(\mathrm{T}_{\mathrm{G}_j, \mu_{k_j}^j, \nu_{k_j}^j} \right)^{p_j} \prod_{j=1}^{n} \left(\partial_{z_j}^{\mu_{k_j}^j} \partial_{\overline{z}_j}^{\nu_{k_j}^j} \right)^{p_j}
\end{aligned}$$

$$\times\ \omega(z_1,\dots,z_n,z_{n+1})$$

$$+\frac{1}{2^{n+1}}\prod_{j=1}^{n+1}\sum_{m_j=0}^{k_j-1}\int\limits_{\partial_0\mathrm{G}^{n+1}}\prod_{j=1}^{n+1}\mathrm{K}_{\mu^j_{m_j+1},\nu^j_{m_j+1}}(z_j-\zeta_j)$$

$$\times\prod_{j=1}^{n+1}\partial_{\zeta_j}^{\mu^j_{m_j}}\partial_{\overline{\zeta}_j}^{\nu^j_{m_j}}\omega(\zeta_1,\dots,\zeta_n,\zeta_{n+1})$$

$$\times\prod_{j=1}^{n+1}d\left[(i\zeta_j)^{\nu^j_{m_j+1}-\nu^j_{m_j}}(-i\overline{\zeta}_j)^{\mu^j_{m_j+1}-\mu^j_{m_j}}\right]$$

$$+\frac{1}{2^{n}}\prod_{j=1}^{n}\sum_{m_j=0}^{k_j-1}\int\limits_{\partial_0\mathrm{G}^{n}}\prod_{j=1}^{n}\mathrm{K}_{\mu^j_{m_j+1},\nu^j_{m_j+1}}(z_j-\zeta_j)$$

$$\times\mathrm{T}_{\mathrm{G}_{n+1},\mu^{n+1}_{k_{n+1}},\nu^{n+1}_{k_{n+1}}}\partial_{z_{n+1}}^{\mu^{n+1}_{k_{n+1}}}\partial_{\overline{z}_{n+1}}^{\nu^{n+1}_{k_{n+1}}}\prod_{j=1}^{n}\partial_{\zeta_j}^{\mu^j_{m_j}}\partial_{\overline{\zeta}_j}^{\nu^j_{m_j}}\omega(\zeta_1,\dots,\zeta_n,z_{n+1})$$

$$\times\prod_{j=1}^{n}d\left[(i\zeta_j)^{\nu^j_{m_j+1}-\nu^j_{m_j}}(-i\overline{\zeta}_j)^{\mu^j_{m_j+1}-\mu^j_{m_j}}\right].$$

In terms of Definition 2

$$\frac{1}{2^{n}}\prod_{j=1}^{n}\sum_{m_j=0}^{k_j-1}\int\limits_{\partial_0\mathrm{G}^{n}}\prod_{j=1}^{n}\mathrm{K}_{\mu^j_{m_j+1},\nu^j_{m_j+1}}(z_j-\zeta_j)$$

$$\times\mathrm{T}_{\mathrm{G}_{n+1},\mu^{n+1}_{k_{n+1}},\nu^{n+1}_{k_{n+1}}}\partial_{z_{n+1}}^{\mu^{n+1}_{k_{n+1}}}\partial_{\overline{z}_{n+1}}^{\nu^{n+1}_{k_{n+1}}}\prod_{j=1}^{n}\partial_{\zeta_j}^{\mu^j_{m_j}}\partial_{\overline{\zeta}_j}^{\nu^j_{m_j}}\omega(\zeta_1,\dots,\zeta_n,z_{n+1})$$

$$\times\prod_{j=1}^{n}d\left[(i\zeta_j)^{\nu^j_{m_j+1}-\nu^j_{m_j}}(-i\overline{\zeta}_j)^{\mu^j_{m_j+1}-\mu^j_{m_j}}\right]$$

$$=\int\limits_{\mathrm{G}_{n+1}}\mathrm{K}_{\mu^{n+1}_{k_{n+1}},\nu^{n+1}_{k_{n+1}}}(z_{n+1}-\zeta_{n+1})$$

$$\times\left\{\frac{1}{2^{n}}\prod_{j=1}^{n}\sum_{m_j=0}^{k_j-1}\int\limits_{\partial_0\mathrm{G}^{n}}\prod_{j=1}^{n}\mathrm{K}_{\mu^j_{m_j+1},\nu^j_{m_j+1}}(z_j-\zeta_j)\right.$$

$$\left.\times\prod_{j=1}^{n}\partial_{\zeta_j}^{\mu^j_{m_j}}\partial_{\overline{\zeta}_j}^{\nu^j_{m_j}}\partial_{\zeta_{n+1}}^{\mu^{n+1}_{k_{n+1}}}\partial_{\overline{\zeta}_{n+1}}^{\nu^{n+1}_{k_{n+1}}}\omega(\zeta_1,\dots,\zeta_n,\zeta_{n+1})\right.$$

$$\times \prod_{j=1}^{n} d\left[(i\zeta_j)^{\nu^j_{m_j+1}-\nu^j_{m_j}}(-i\overline{\zeta}_j)^{\mu^j_{m_j+1}-\mu^j_{m_j}}\right]\Bigg\} d\xi_{n+1}d\eta_{n+1}$$

holds. Representing

$$\partial^{\mu^{n+1}_{k_{n+1}}}_{\zeta_{n+1}}\partial^{\nu^{n+1}_{k_{n+1}}}_{\overline{\zeta}_{n+1}}\omega(z_1,\dots,z_n,\zeta_{n+1})$$

with the help of (2) in G^n shows

$$\begin{aligned}
&\int\limits_{\mathrm{G}_{n+1}} \mathrm{K}_{\mu^{n+1}_{k_{n+1}},\nu^{n+1}_{k_{n+1}}}(z_{n+1}-\zeta_{n+1})\\
&\qquad\times\Bigg\{\frac{1}{2^n}\prod_{j=1}^{n}\sum_{m_j=0}^{k_j-1}\int\limits_{\partial_0\mathrm{G}^n}\prod_{j=1}^{n}\mathrm{K}_{\mu^j_{m_j+1},\nu^j_{m_j+1}}(z_j-\zeta_j)\\
&\times\prod_{j=1}^{n}\partial^{\mu^j_{m_j}}_{\zeta_j}\partial^{\nu^j_{m_j}}_{\overline{\zeta}_j}\partial^{\mu^{n+1}_{k_{n+1}}}_{\zeta_{n+1}}\partial^{\nu^{n+1}_{k_{n+1}}}_{\overline{\zeta}_{n+1}}\omega(\zeta_1,\dots,\zeta_n,\zeta_{n+1})\\
&\times\prod_{j=1}^{n}d\left[(i\zeta_j)^{\nu^j_{m_j+1}-\nu^j_{m_j}}(-i\overline{\zeta}_j)^{\mu^j_{m_j+1}-\mu^j_{m_j}}\right]\Bigg\}d\xi_{n+1}d\eta_{n+1}\\
&=\int\limits_{\mathrm{G}_{n+1}}\mathrm{K}_{\mu^{n+1}_{k_{n+1}},\nu^{n+1}_{k_{n+1}}}(z_{n+1}-\zeta_{n+1})\Bigg\{\partial^{\mu^{n+1}_{k_{n+1}}}_{\zeta_{n+1}}\partial^{\nu^{n+1}_{k_{n+1}}}_{\overline{\zeta}_{n+1}}\omega(z_1,\dots,z_n,\zeta_{n+1})\\
&-\sum_{\substack{(p_1,\dots,p_n)\in\{0,1\}^n\\(p_1,\dots,p_n)\neq(0,\dots,0)}}(-1)^{1+\sum_{j=1}^n p_j}\prod_{j=1}^{n}\left(\mathrm{T}_{\mathrm{G}_j,\mu^j_{k_j},\nu^j_{k_j}}\right)^{p_j}\\
&\times\prod_{j=1}^{n}\left(\partial^{\mu^j_{k_j}}_{z_j}\partial^{\nu^j_{k_j}}_{\overline{z}_j}\right)^{p_j}\partial^{\mu^{n+1}_{k_{n+1}}}_{\zeta_{n+1}}\partial^{\nu^{n+1}_{k_{n+1}}}_{\overline{\zeta}_{n+1}}\omega(z_1,\dots,z_n,\zeta_{n+1})\Bigg\}d\xi_{n+1}d\eta_{n+1}\\
&=\int\limits_{\mathrm{G}_{n+1}}\mathrm{K}_{\mu^{n+1}_{k_{n+1}},\nu^{n+1}_{k_{n+1}}}(z_{n+1}-\zeta_{n+1})\\
&\times\partial^{\mu^{n+1}_{k_{n+1}}}_{\zeta_{n+1}}\partial^{\nu^{n+1}_{k_{n+1}}}_{\overline{\zeta}_{n+1}}\omega(z_1,\dots,z_n,\zeta_{n+1})d\xi_{n+1}d\eta_{n+1}\\
&-\int\limits_{\mathrm{G}_{n+1}}\mathrm{K}_{\mu^{n+1}_{k_{n+1}},\nu^{n+1}_{k_{n+1}}}(z_{n+1}-\zeta_{n+1})\sum_{\substack{(p_1,\dots,p_n)\in\{0,1\}^n\\(p_1,\dots,p_n)\neq(0,\dots,0)}}(-1)^{1+\sum_{j=1}^n p_j}
\end{aligned}$$

$$
\begin{aligned}
&\times \prod_{j=1}^{n} \left(\mathrm{T}_{\mathrm{G}_j,\mu_{k_j}^j,\nu_{k_j}^j} \right)^{p_j} \prod_{j=1}^{n} \left(\partial_{z_j}^{\mu_{k_j}^j} \partial_{\overline{z}_j}^{\nu_{k_j}^j} \right)^{p_j} \\
&\times \partial_{\zeta_{n+1}}^{\mu_{k_{n+1}}^{n+1}} \partial_{\overline{\zeta}_{n+1}}^{\nu_{k_{n+1}}^{n+1}} \omega(z_1,\ldots,z_n,\zeta_{n+1}) d\xi_{n+1} d\eta_{n+1} \\
&= \mathrm{T}_{\mathrm{G}_{n+1},\mu_{k_{n+1}}^{n+1},\nu_{k_{n+1}}^{n+1}} \partial_{z_{n+1}}^{\mu_{k_{n+1}}^{n+1}} \partial_{\overline{z}_{n+1}}^{\nu_{k_{n+1}}^{n+1}} \omega(z_1,\ldots,z_n,z_{n+1}) \\
&+ \sum_{\substack{(p_1,\ldots,p_n,p_{n+1})\in\{0,1\}^{n+1} \\ (p_1,\ldots,p_n)\neq(0,\ldots,0) \\ p_{n+1}=1}} (-1)^{\sum_{j=1}^{n} p_j} \\
&\times \prod_{j=1}^{n+1} \left(\mathrm{T}_{\mathrm{G}_j,\mu_{k_j}^j,\nu_{k_j}^j} \right)^{p_j} \prod_{j=1}^{n+1} \left(\partial_{z_j}^{\mu_{k_j}^j} \partial_{\overline{z}_j}^{\nu_{k_j}^j} \right)^{p_j} \omega(z_1,\ldots,z_n,z_{n+1}).
\end{aligned}
$$

Since for $p_{n+1} = 1$

$$(-1)^{\sum_{j=1}^{n} p_j} = (-1)^{1+p_{n+1}+\sum_{j=1}^{n} p_j} = (-1)^{1+\sum_{j=1}^{n+1} p_j},$$

the proof is finished. □

3 An application

To give an application of the representation formula (2) for $n \in \mathbb{N}$ the following special chains of bi–indices are considered:

$$
\begin{array}{rclrclrcl}
(\mu_0^1,\nu_0^1) & = & (0,0), & (\mu_1^1,\nu_1^1) & = & (0,1), & (\mu_2^1,\nu_2^1) & = & (1,1), \\
(\mu_0^j,\nu_0^j) & = & (0,0), & (\mu_1^j,\nu_1^j) & = & (1,0), & (\mu_2^j,\nu_2^j) & = & (1,1), \\
 & & & & & \leq j \leq n. & & &
\end{array} \tag{6}
$$

Furthermore let $\mathrm{G}_j \subset \mathbb{C}$, $1 \leq j \leq n$, be bounded domains with smooth boundaries and

$$\omega \in \mathrm{C}^2(\mathrm{G}^n;\mathbb{C}) \cap \mathrm{C}^1(\partial_0 \mathrm{G}^n \cup \mathrm{G}^n;\mathbb{C}),$$

where the preceding symbol denotes a function $\omega : \partial_0 \mathrm{G}^n \cup \mathrm{G}^n \to \mathbb{C}$ with continuous

$$\prod_{j=1}^{n} \left(\partial_{z_j}\right)^{p_1^j} \left(\partial_{\overline{z}_j}\right)^{p_2^j} \omega \quad \text{for all} \quad \left((p_1^1,p_2^1),\ldots,(p_1^n,p_2^n)\right) \in \left\{\{0,1\}^2\right\}^n$$

in G^n and continuous

$$\prod_{j=1}^{n} \left(\partial_{z_j}\right)^{p_1^j} \left(\partial_{\overline{z}_j}\right)^{p_2^j} \omega \quad \text{for all} \quad \left((p_1^1, p_2^1), \ldots, (p_1^n, p_2^n)\right) \in \{(0,1),(1,0)\}^n$$

in $\partial_0\mathrm{G}^n \cup \mathrm{G}^n$. These assumptions together with (6) show that ω is a solution of

$$\begin{aligned} \partial_{z_1}^{\mu_{m_1}^1} \partial_{\overline{z}_1}^{\nu_{m_1}^1} \omega &= f_{m_1} \in \mathrm{L}_1(\mathrm{G}^n;\mathbb{C}), \quad 0 \le m_1 \le 2, \ldots, \\ \prod_{\rho=1}^{n} \partial_{z_\rho}^{\mu_{m_\rho}^\rho} \partial_{\overline{z}_\rho}^{\nu_{m_\rho}^\rho} \omega &= f_{m_1,\ldots,m_n} \in \mathrm{L}_1(\mathrm{G}^n;\mathbb{C}), \quad 0 \le m_j \le 2, \\ & \qquad\qquad 1 \le j \le n, \quad 2 \le n, \end{aligned}$$

in G,

$$\begin{aligned} \partial_{z_1}^{\mu_{k_1-1}^1} \partial_{\overline{z}_1}^{\nu_{k_1-1}^1} \omega &= \varphi_{k_1-1} \in \mathrm{C}(\partial_0\mathrm{G}^n;\mathbb{C}), \ldots, \\ \prod_{\rho=1}^{n} \partial_{z_\rho}^{\mu_{k_\rho-1}^\rho} \partial_{\overline{z}_\rho}^{\nu_{k_\rho-1}^\rho} \omega &= \varphi_{k_1-1,\ldots,k_{n-1}-1} \in \mathrm{C}(\partial_0\mathrm{G}^n;\mathbb{C}), \end{aligned}$$

on $\partial_0\mathrm{G}^n$ with the compatibility conditions

$$\begin{aligned} \partial_{z_2}^{\mu_2^2} \partial_{\overline{z}_2}^{\nu_2^2} f_{m_1} &= f_{m_1,2}, \quad 0 \le m_1 \le 1, \ldots, \\ \partial_{z_n}^{\mu_2^n} \partial_{\overline{z}_n}^{\nu_2^n} f_{m_1,\ldots,m_{n-1}} &= f_{m_1,\ldots,m_{n-1},2}, \quad 0 \le m_j \le 1, \quad 1 \le j \le n-1. \end{aligned}$$

Hence ω meets the requirements of Theorem 2 and is therefore representable as

$$\begin{aligned} \omega(z) &= \sum_{\substack{(p_1,\ldots,p_n)\in\{0,1\}^n \\ (p_1,\ldots,p_n)\neq(0,\ldots,0)}} (-1)^{1+\sum_{j=1}^n p_j} \prod_{j=1}^{n} \left(\mathrm{T}_{\mathrm{G}_j,1,1}\right)^{p_j} \prod_{j=1}^{n} \left(\partial_{z_j}\partial_{\overline{z}_j}\right)^{p_j} \omega(z) \\ &\quad + \frac{1}{2^n} \prod_{\tau=1}^{n} \sum_{m_\tau=0}^{1} \int_{\partial_0\mathrm{G}^n} \prod_{j=1}^{n} \mathrm{K}_{\mu_{m_j+1}^j,\nu_{m_j+1}^j}(z_j-\zeta_j) \prod_{j=1}^{n} \partial_{\zeta_j}^{\mu_{m_j}^j} \partial_{\overline{\zeta}_j}^{\nu_{m_j}^j} \omega(\zeta) \\ &\qquad \times \prod_{j=1}^{n} d\left[(i\zeta_j)^{\nu_{m_j+1}^j-\nu_{m_j}^j} (-i\overline{\zeta}_j)^{\mu_{m_j+1}^j-\mu_{m_j}^j}\right]. \end{aligned}$$

This can be transformed into a more explicit form. A simple induction shows first

$$\sum_{\substack{(p_1,\dots,p_n)\in\{0,1\}^n\\(p_1,\dots,p_n)\neq(0,\dots,0)}} (-1)^{1+\sum_{j=1}^n p_j} \prod_{j=1}^n \left(\mathrm{T}_{\mathrm{G}_j,1,1}\right)^{p_j} \prod_{j=1}^n \left(\partial_{z_j}\partial_{\overline{z}_j}\right)^{p_j} \omega(z)$$

$$= \sum_{\tau=1}^n (-1)^{\tau+1} \sum_{1\le\nu_1<\ldots<\nu_\tau\le n} \frac{1}{\pi^\tau} \int\limits_{\mathrm{G}_{\nu_1}} \dots \int\limits_{\mathrm{G}_{\nu_\tau}} \omega_{\zeta_{\nu_1}\overline{\zeta}_{\nu_1}\dots\zeta_{\nu_\tau}\overline{\zeta}_{\nu_\tau}}(\zeta) \quad (7)$$

$$\times \prod_{\rho=1}^\tau \log|\zeta_{\nu_\rho} - z_{\nu_\rho}|^2 \prod_{\rho=1}^\tau d\xi_{\nu_\rho} d\eta_{\nu_\rho},$$

where in the last expression the ζ_{ν_j}'s, which are not integrated upon are understood as z_{ν_j}'s. With

Definition 3 *For*

$(p_1,\dots,p_n)\in\{0,1\}^n,\ \zeta=(\zeta_1,\dots\zeta_n)\in\mathbb{C}^n$ *and* $z=(z_1,\dots,z_n)\in\mathbb{C}^n$

let

$$\begin{aligned}
\mathrm{D}^{(p_1,\dots,p_n)} &:= \frac{\partial^{\sum_{j=1}^n p_j}}{\partial\overline{\zeta}_1^{p_1}\partial\zeta_2^{p_2}\dots\partial\zeta_n^{p_n}},\\
k(z_1,p_1) &:= \begin{cases} z_1^{-1}, & p_1=0,\\ \log|z_1|^2, & p_1=1,\end{cases}\\
k(z_j,p_j) &:= \begin{cases} \overline{z}_j^{-1}, & p_j=0,\\ \log|z_j|^2, & p_j=1,\end{cases} \qquad 2\le j\le n,\\
d(\zeta_1^{p_1}) &:= \begin{cases} d\zeta_1, & p_1=0,\\ d\overline{\zeta}_1, & p_1=1,\end{cases}\\
d(\zeta_j^{p_j}) &:= \begin{cases} d\overline{\zeta}_j, & p_j=0,\\ d\zeta_j, & p_j=1,\end{cases} \qquad 2\le j\le n.
\end{aligned}$$

and

$$\begin{aligned}
\pi\mathrm{K}_{\mu^1_{m_1+1},\nu^1_{m_1+1}}(z_1-\zeta_1) &= \begin{cases} (z_1-\zeta_1)^{-1}, & m_1=0,\\ \log|z_1-\zeta_1|^2, & m_1=1,\end{cases}\\
\pi\mathrm{K}_{\mu^j_{m_j+1},\nu^j_{m_j+1}}(z_j-\zeta_j) &= \begin{cases} (\overline{z_j-\zeta_j})^{-1}, & m_j=0,\\ \log|z_j-\zeta_j|^2, & m_j=1,\end{cases}, \quad 2\le j\le n,
\end{aligned}$$

where the kernels

$$\mathrm{K}_{\mu^j_{m_j+1},\nu^j_{m_j+1}}(z_j-\zeta_j)$$

for $j = 1$ in the case of $m_1 = 0$ correspond with $d\zeta_1$ and in the case of $m_1 = 1$ with $d\overline{\zeta}_1$ as well as for $j \geq 2$ in the case of $m_j = 0$ with $d\overline{\zeta}_j$ and in the case of $m_j = 1$ with $d\zeta_j$, and the fact that every sum of

$$\prod_{\tau=1}^{n} \sum_{m_\tau=0}^{1} \tag{8}$$

posseses exactly two summands, so that in total the number of summands of (8) equals the number of summands of

$$\sum_{(p_1,\ldots,p_n)\in\{0,1\}^n},$$

it holds

$$\begin{aligned}
&\frac{1}{2^n} \prod_{\tau=1}^{n} \sum_{m_\tau=0}^{1} \int\limits_{\partial_0 \mathrm{G}^n} \prod_{j=1}^{n} \mathrm{K}_{\mu^j_{m_j+1},\nu^j_{m_j+1}}(z_j - \zeta_j) \prod_{j=1}^{n} \partial_{\zeta_j}^{\mu^j_{m_j}} \partial_{\overline{\zeta}_j}^{\nu^j_{m_j}} \omega(\zeta) \\
&\quad \times \prod_{j=1}^{n} d\left[(i\zeta_j)^{\nu^j_{m_j+1} - \nu^j_{m_j}} (-i\overline{\zeta}_j)^{\mu^j_{m_j+1} - \mu^j_{m_j}} \right] \\
&\quad = \frac{(-1)^{n+1}}{(2\pi i)^n} \int\limits_{\partial_0 \mathrm{G}^n} \sum_{(p_1,\ldots,p_n)\in\{0,1\}^n} \left(\mathrm{D}^{(p_1,\ldots,p_n)} \omega(\zeta) \right) \prod_{j=1}^{n} k(\zeta_j - z_j, p_j) \\
&\quad \times \prod_{j=1}^{n} d(\zeta_j^{p_j}).
\end{aligned}$$

Hence in connection with (7) the following theorem holds.

Theorem 3 *Every* $\omega \in \mathrm{C}^2(\mathrm{G}^n;\mathbb{C}) \cap \mathrm{C}^1(\partial_0 \mathrm{G}^n \cup \mathrm{G}^n;\mathbb{C})$, $\mathrm{G}_j \subset \mathbb{C}$, $1 \leq j \leq n$, *bounded domains with smooth boundaries, is for* $z \in \mathrm{G}^n$ *representable as*

$$\begin{aligned}
\omega(z) &= \sum_{\tau=1}^{n} (-1)^{\tau+1} \sum_{1\leq\nu_1<\ldots<\nu_\tau\leq n} \frac{1}{\pi^\tau} \int\limits_{\mathrm{G}_{\nu_1}} \cdots \int\limits_{\mathrm{G}_{\nu_\tau}} \omega_{\zeta_{\nu_1}\overline{\zeta}_{\nu_1}\ldots\zeta_{\nu_\tau}\overline{\zeta}_{\nu_\tau}}(\zeta) \\
&\quad \times \prod_{\rho=1}^{\tau} \log|\zeta_{\nu_\rho} - z_{\nu_\rho}|^2 \prod_{\rho=1}^{\tau} d\xi_{\nu_\rho} d\eta_{\nu_\rho} \\
&\quad + \frac{(-1)^{n+1}}{(2\pi i)^n} \int\limits_{\partial_0 \mathrm{G}^n} \sum_{(p_1,\ldots,p_n)\in\{0,1\}^n} \left(\mathrm{D}^{(p_1,\ldots,p_n)} \omega(\zeta) \right) \prod_{j=1}^{n} k(\zeta_j - z_j, p_j) \\
&\quad \times \prod_{j=1}^{n} d(\zeta_j^{p_j}).
\end{aligned}$$

Remark The statement of Theorem 3 corresponds to a $(n-1)$–times repeated Iteration of

$$\begin{aligned}\omega(z) &= \frac{1}{2\pi i}\int\limits_{\partial \mathrm{G}} \omega(\zeta)\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int\limits_{\partial \mathrm{G}} \log|\zeta - z|^2 \omega_{\overline{\zeta}}(\zeta)d\overline{\zeta} \\ &\quad + \frac{1}{\pi}\int\limits_{\mathrm{G}} \log|\zeta - z|^2 \omega_{\zeta\overline{\zeta}}(\zeta)d\xi d\eta\end{aligned}$$

with

$$\begin{aligned}\omega(z) &= -\frac{1}{2\pi i}\int\limits_{\partial \mathrm{G}} \omega(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{2\pi i}\int\limits_{\partial \mathrm{G}} \log|\zeta - z|^2 \omega_{\zeta}(\zeta)d\zeta \\ &\quad + \frac{1}{\pi}\int\limits_{\mathrm{G}} \log|\zeta - z|^2 \omega_{\zeta\overline{\zeta}}(\zeta)d\xi d\eta\end{aligned}$$

in G^n (see [3]).

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Higher order Green function in the upper half plane

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Abstract

The polyharmonic Green function is defined for the upper half plane of the complex plane and its properties are developed.
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1 Introduction

The Green function for an arbitrary entire power of the Laplace operator was given in explicit form in [1] for discs. They were used e.g. in [5,6] to treat boundary value problems for polyharmonic functions in plane domains. In [3] this Green function was used to modify higher order Cauchy-Pompeiu representations, see e.g. [2], by replacing the higher order Cauchy kernel by proper derivatives of a related polyharmonic Green function.

Here the higher order Green function is defined for the upper half plane and its properties proved which later will be used to develop and modify higher order Cauchy-Pompeiu formulas for this particular unbounded domain.

2 Basic properties of the Green function

The Green function for the upper half plane $\mathbb{H}$ of the complex plane $\mathbb{C}$ has the following properties. For any fixed $\zeta \in \mathbb{H}$ as a function of z

(1) $G(z,\zeta)$ is harmonic in $\mathbb{H}\backslash\{\zeta\}$,

(2) $G(z,\zeta) + \log|\zeta - z|$ is harmonic in $\mathbb{H}$,

(3) $\lim_{z\to t} G(z,\zeta) = 0$ for all $t \in \partial\mathbb{H}$.

Green functions exist for other domains than just for the upper half plane. The existence is related to the solvability of the Dirichlet problem for harmonic functions in the domain.

Twice the Green function of the half plane $\mathbb{H} = \{z : z \in \mathbb{C}, Imz > 0\}$ of the complex plane $\mathbb{C}$ is

$$G_1(z,\zeta) = \log|\frac{\bar{\zeta} - z}{\zeta - z}|^2,$$

where $z, \zeta \in \mathbb{H}, z \neq \zeta$. For convenience G_1 is used instead of G and in the sequal G_1 is called the Green function of $\mathbb{H}$.

Definition 1 *Let for* $1 \leqslant n$ *and* $z, \zeta \in \mathbb{H}$, $z \neq \zeta$

$$(n-1)!^2 G_n(z,\zeta) = |\zeta - z|^{2(n-1)} \log|\frac{\bar{\zeta} - z}{\zeta - z}|^2$$

$$+ \sum_{\nu=1}^{n-1} \frac{1}{\nu} |\zeta - z|^{2(n-1-\nu)} (\zeta - \bar{\zeta})^{\nu} (z - \bar{z})^{\nu}. \qquad (1)$$

G_n is called the Green function of order n for $\mathbb{H}$.

Lemma 1 *The function G_n satisfies $G_n(z,\zeta) = G_n(\zeta,z)$ and*

$$\partial_z G_n(z,\zeta) = -\frac{(\zeta-z)^{n-2}\overline{(\zeta-z)}^{n-1}}{(n-1)!(n-2)!}\log|\frac{\bar\zeta - z}{\zeta - z}|^2$$

$$-\frac{1}{(n-1)!(n-2)!}\sum_{\nu=1}^{n-2}\frac{1}{\nu}(\zeta-z)^{n-2-\nu}\overline{(\zeta-z)}^{n-1-\nu}(\zeta-\bar\zeta)^\nu(z-\bar z)^\nu$$

$$-\frac{1}{(n-1)!^2}\frac{(\zeta-\bar\zeta)^{n-1}(z-\bar z)^{n-1}}{\bar\zeta - z}, \tag{2}$$

$$\partial_z\partial_{\bar z}G_n(z,\zeta) = G_{n-1}(z,\zeta) + \frac{(\zeta-\bar\zeta)^{n-1}(z-\bar z)^{n-2}}{(n-1)(n-2)}(\frac{1}{\zeta-z} - \frac{1}{\zeta-\bar z}) \tag{3}$$

for $z,\zeta\in\mathbb{H}$, $z\neq\zeta$, and for $0\leqslant\rho<n$

$$\partial_z^\rho G_n(z,\zeta) = \frac{(-1)^\rho}{(n-1-\rho)!(n-1)!}(\zeta-z)^{n-\rho-1}\overline{(\zeta-z)}^{n-1}\log|\frac{\bar\zeta-z}{\zeta-z}|^2$$

$$+\sum_{\nu=0}^{\rho-1}\binom{\rho}{\nu}(-1)^\nu\frac{(\zeta-z)^{n-1-\nu}(\overline{\zeta-z})^{n-1}}{(n-1-\nu)!(n-1)!}(\rho-\nu-1)!$$

$$\times(\frac{1}{(\zeta-z)^{\rho-\nu}} - \frac{1}{(\bar\zeta-z)^{\rho-z}}) + \frac{1}{(n-1)!^2}\sum_{\nu=1}^{n-1}\frac{1}{\nu}(\zeta-\bar\zeta)^\nu$$

$$\times\sum_{\sigma=max\{0,\rho-\nu\}}^{min\{\rho,n-1-\nu\}}\binom{\rho}{\sigma}\frac{(n-1-\nu)!}{(n-1-\nu-\sigma)!}(\zeta-z)^{n-1-\nu-\sigma}(\overline{\zeta-z})^{n-1-\nu}$$

$$\times\frac{\nu!}{(\nu-\rho+\sigma)!}(z-\bar z)^{\nu-\rho+\sigma}. \tag{4}$$

Proof For $\rho=0$ obviously, (4) holds. Applying the Leibniz rule for $1\leqslant\rho\leqslant n-1$ by differentiating

$$G_n(z,\zeta) = \frac{1}{(n-1)!^2}|\zeta-z|^{2(n-1)}\log|\frac{\bar\zeta-z}{\zeta-z}|^2$$

$$+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{n-1}\frac{1}{\nu}|\zeta-z|^{2(n-1-\nu)}(\zeta-\bar\zeta)^\nu(z-\bar z)^\nu,$$

one gets

$$\partial_z^{\rho+1}G_n(z,\zeta)=\frac{(-1)^{\rho+1}}{(n-2-\rho)!(n-1)!}(\zeta-z)^{n-\rho-2}\overline{(\zeta-z)}^{n-1}\log|\frac{\bar{\zeta}-z}{\zeta-z}|^2$$

$$+\frac{(-1)^{\rho}}{(n-1-\rho)!(n-1)!}(\zeta-z)^{n-\rho-1}\overline{(\zeta-z)^{n-1}}(\frac{1}{\zeta-z}-\frac{1}{\bar{\zeta}-z})$$

$$+\sum_{\nu=0}^{\rho-1}\binom{\rho}{\nu}(-1)^{\nu+1}\frac{(\zeta-z)^{n-2-\nu}\overline{(\zeta-z)}^{n-1}}{(n-2-\nu)!(n-1)!}(\rho-\nu-1)!$$

$$\times\left(\frac{1}{(\zeta-z)^{\rho-\nu}}-\frac{1}{(\bar{\zeta}-z)^{\rho-\nu}}\right)$$

$$+\sum_{\nu=0}^{\rho-1}\binom{\rho}{\nu}(-1)^{\nu}\frac{(\zeta-z)^{n-1-\nu}\overline{(\zeta-z)}^{n-1}}{(n-1-\nu)!(n-1)!}(\rho-\nu)!$$

$$\times\left(\frac{1}{(\zeta-z)^{\rho-\nu+1}}-\frac{1}{(\bar{\zeta}-z)^{\rho-\nu+1}}\right)$$

$$+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{n-1}\frac{1}{\nu}(\zeta-\bar{\zeta})^{\nu}\sum_{\sigma=max\{0,\rho-\nu\}}^{min\{\rho,n-2-\nu\}}(-1)^{\sigma+1}\binom{\rho}{\sigma}\frac{(n-1-\nu)!}{(n-2-\nu-\sigma)!}$$

$$\times(\zeta-z)^{n-2-\nu-\sigma}\overline{(\zeta-z)}^{n-1-\nu}\frac{\nu!}{(\nu-\rho+\sigma)!}(z-\bar{z})^{\nu-\rho+\sigma}$$

$$+\frac{1}{(n-1)^2}\sum_{\nu=1}^{n-1}\frac{1}{\nu}(\zeta-\bar{\zeta})^{\nu}\sum_{\sigma=max\{0,\rho-\nu+1\}}^{min\{\rho,n-1-\nu\}}(-1)^{\sigma}\binom{\rho}{\sigma}\frac{(n-1-\nu)!}{(n-1-\nu-\sigma)!}$$

$$\times(\zeta-z)^{n-1-\nu-\sigma}\overline{(\zeta-z)}^{n-1-\nu}\frac{\nu!}{(\nu-\rho-1-\sigma)!}(z-\bar{z})^{\nu-\rho-1-\sigma}$$

$$=\frac{(-1)^{\rho+1}}{(n-1-\rho-1)!(n-1)!}(\zeta-z)^{n-1-\rho-1}\overline{(\zeta-z)}^{n-1}\log|\frac{\bar{\zeta}-z}{\zeta-z}|^2$$

$$+\frac{(-1)^{\rho}}{(n-1-\rho)!(n-1)!}(\zeta-z)^{n-\rho-1}\overline{(\zeta-z)}^{n-1}\left(\frac{1}{\zeta-z}-\frac{1}{\bar{\zeta}-z}\right)$$

$$+\sum_{\nu=1}^{\rho}\binom{\rho}{\nu-1}(-1)^{\nu}\frac{(\zeta-z)^{n-1-\nu}\overline{(\zeta-z)}^{n-1}}{(n-1-\nu)!(n-1)!}(\rho-\nu)!$$

$$\times\left(\frac{1}{(\zeta-z)^{\rho-\nu+1}}-\frac{1}{(\bar{\zeta}-z)^{\rho-\nu+1}}\right)$$

$$+\sum_{\nu=0}^{\rho-1}\binom{\rho}{\nu}(-1)^{\nu}\frac{(\zeta-z)^{n-1-\nu}\overline{(\zeta-z)}^{n-1}}{(n-1-\nu)!(n-1)!}(\rho-\nu)!$$

$$\times\left(\frac{1}{(\zeta-z)^{\rho+2-\nu}}-\frac{1}{(\bar{\zeta}-z)^{\rho+1-\nu}}\right)$$

$$+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{n-1}\frac{1}{\nu}(\zeta-\bar{\zeta})^{\nu}\sum_{\sigma=max\{1,\rho-\nu+1\}}^{min\{\rho+1,n-1-\nu\}}(-1)^{\sigma}\binom{\rho}{\sigma-1}\frac{(n-1-\nu)!}{(n-1-\nu-\sigma)!}$$

$$\times(\zeta-z)^{n-1-\nu-\sigma}\overline{(\zeta-z)}^{n-1-\nu}\frac{\nu!}{(\nu-\rho+\sigma-1)!}(z-\bar{z})^{\nu-\rho+\sigma-1}$$

$$+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{n-1}\frac{1}{\nu}(\zeta-\bar{\zeta})^{\nu}\sum_{\sigma=max\{0,\rho-\nu+1\}}^{min\{\rho,n-1-\nu\}}(-1)^{\sigma}\binom{\rho}{\sigma}\frac{(n-1-\nu)!}{(n-1-\nu-\sigma)!}$$

$$\times(\zeta-z)^{n-1-\nu-\sigma}\overline{(\zeta-z)}^{n-1-\nu}\frac{\nu!}{(\nu-\rho-1-\sigma)!}(z-\bar{z})^{\nu+\rho-1-\sigma}$$

$$=\frac{(-1)^{\rho+1}}{(n-1-\rho-1)!(n-1)!}(\zeta-z)^{n-1-\rho-1}\overline{(\zeta-z)}^{n-1}\log|\frac{\bar{\zeta}-z}{\zeta-z}|^2$$

$$+\sum_{\nu=0}^{\rho}(-1)^{\nu}\binom{\rho+1}{\nu}\frac{(\zeta-z)^{n-1-\nu}\overline{(\zeta-z)}^{n-1}}{(n-1-\nu)!(n-1)!}(\rho-\nu)!$$

$$\times(\frac{1}{(\zeta-z)^{\rho+1-\nu}}-\frac{1}{(\bar{\zeta}-z)^{\rho+1-\nu}})+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{n-1}\frac{1}{\nu}(\zeta-\bar{\zeta})^{\nu}$$

$$\times\sum_{\sigma=max\{0,\rho-\nu+1\}}^{min\{\rho+1,n-1-\nu\}}(-1)^{\sigma}\binom{\rho+1}{\sigma}\frac{(n-1-\nu)!}{(n-1-\nu-\sigma)!}$$

$$\times(\zeta-z)^{n-1-\nu-\sigma}\overline{(\zeta-z)}^{n-1-\nu}\frac{\nu!}{(\nu-\rho-1-\sigma)!}(z-\bar{z})^{\nu-\rho-1-\sigma}.$$

This is (4) for $\rho+1$ rather than for ρ. Thus (4) is proved by induction. Applying (4) for $\rho=n-1$ gives

$$\partial_z^{n-1}G_n(z,\zeta)=\frac{(-1)^{n-1}}{(n-1)!}\overline{(\zeta-z)}^{n-1}\log|\frac{\bar{\zeta}-z}{\zeta-z}|^2+\sum_{\nu=0}^{n-2}\binom{n-1}{\nu}$$

$$\times(-1)^{\nu}\frac{(\zeta-z)^{n-1-\nu}\overline{(\zeta-z)}^{n-1}}{(n-1-\nu)!(n-1)!}(n-2-\nu)!$$

$$\times\left(\frac{1}{(\zeta-z)^{n-1-\nu}}-\frac{1}{(\bar{\zeta}-z)^{n-1-\nu}}\right)$$

$$+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{n-1}\frac{1}{\nu}(\zeta-\bar{\zeta})^{\nu}(-1)^{n-1-\nu}\binom{n-1}{n-1-\nu}$$

$$\times\frac{(n-1-\nu)!\nu!}{\nu!}\overline{(\zeta-z)}^{n-1-\nu}$$

$$=\frac{(-1)^{n-1}}{(n-1)!}\overline{(\zeta-z)}^{n-1}\log|\frac{\bar{\zeta}-z}{\zeta-z}|^2+\sum_{\nu=0}^{n-2}(-1)^{\nu}\binom{n-1}{\nu}\frac{(\zeta-z)^{n-1-\nu}}{n-1-\nu}$$

$$\times\frac{\overline{(\zeta-z)}^{n-1}}{(n-1)!}(\frac{1}{(\zeta-z)^{n-1-\nu}}-\frac{1}{(\bar{\zeta}-z)^{n-1-\nu}})$$

$$+\frac{1}{(n-1)!}\sum_{\nu=1}^{n-1}\frac{(-1)^{n-1-\nu}}{\nu}(\zeta-\bar{\zeta})^{\nu}\overline{(\zeta-z)}^{n-1-\nu}$$

$$=\frac{(-1)^{n-1}}{(n-1)!}\overline{(\zeta-z)}^{n-1}\log|\frac{\bar{\zeta}-z}{\zeta-z}|^2+\frac{\overline{(\zeta-z)}^{n-1}}{(n-1)!}\sum_{\nu=0}^{n-2}\frac{(-1)^{\nu}}{n-1-\nu}$$

$$\times\binom{n-1}{\nu}(1-(\frac{\zeta-z}{\bar{\zeta}-z})^{n-1-\nu})+\frac{\overline{(\zeta-z)}^{n-1}}{(n-1)!}\sum_{\nu=1}^{n-1}\frac{(-1)^{n-1-\nu}}{\nu}\left(\frac{\zeta-\bar{\zeta}}{\overline{\bar{\zeta}-z}}\right)^{\nu}.$$

Corollary 1 *For any* $n\in\mathbb{N}$

$$\partial_z^n G_n(z,\zeta)=\frac{(-1)^{n-1}}{(n-1)!}(\frac{\zeta-\bar{\zeta}}{\bar{\zeta}-z})^n\frac{\overline{(\zeta-z)}^{n-1}}{\zeta-z}. \tag{5}$$

Proof Differentiating the preceding formula shows

$$\partial_z^n G_n(z,\zeta)=\frac{(-1)^{n-1}}{(n-1)!}\overline{(\zeta-z)}^{n-1}\left(\frac{1}{\zeta-z}-\frac{1}{\bar{\zeta}-z}\right)$$

$$+\sum_{\nu=0}^{n-2}(-1)^{\nu+1}\binom{n-1}{\nu}$$

$$\times(\zeta-z)^{n-2-\nu}\frac{\overline{(\zeta-z)}^{n-1}}{(n-1)!}\left(\frac{1}{(\zeta-z)^{n-1-\nu}}-\frac{1}{(\bar{\zeta}-z)^{n-1-\nu}}\right)$$

$$+\sum_{\nu=0}^{n-2}(-1)^{\nu}\binom{n-1}{\nu}$$

$$\times(\zeta-z)^{n-1-\nu}\frac{\overline{(\zeta-z)}^{n-1}}{(n-1)!}\left(\frac{1}{(\zeta-z)^{n-\nu}}-\frac{1}{(\bar{\zeta}-z)^{n-\nu}}\right)$$

$$
\begin{aligned}
&= \sum_{\nu=0}^{n-1} (-1)^{\nu} \binom{n}{\nu} (\zeta - z)^{n-1-\nu} \\
&\times \frac{\overline{(\zeta - z)}^{n-1}}{(n-1)!} \left(\frac{1}{(\zeta - z)^{n-\nu}} - \frac{1}{(\bar{\zeta} - z)^{n-\nu}} \right) \\
&= \frac{\overline{(\zeta - z)}^{n-1}}{(n-1)!} \sum_{\nu=0}^{n-1} (-1)^{\nu} \binom{n}{\nu} \left(1 - \left(\frac{\zeta - z}{\bar{\zeta} - z} \right)^{n-\nu} \right) \frac{1}{\zeta - z} \\
&= \frac{\overline{(\zeta - z)}^{n-1}}{(n-1)!(\zeta - z)} \left[\sum_{\nu=0}^{n-1} (-1)^{\nu} \binom{n}{\nu} - \sum_{\nu=0}^{n-1} (-1)^{\nu} \binom{n}{\nu} \left(\frac{\zeta - z}{\bar{\zeta} - z} \right)^{n} \right] \\
&= \frac{\overline{(\zeta - z)}^{n-1}}{(n-1)!} \frac{1}{\zeta - z} \left[(1-1)^n - (-1)^n - \left(-1 + \frac{\zeta - z}{\bar{\zeta} - z} \right)^{n} + (-1)^n \right] \\
&= \frac{\overline{(\zeta - z)}^{n-1} (-1)^{n-1}}{(n-1)!(\zeta - z)} \left(\frac{\zeta - z}{\bar{\zeta} - z} - 1 \right)^{n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{(\zeta - \bar{\zeta})^{n}}{|\zeta - z|^2} \left(\frac{\overline{(\zeta - z)}}{\bar{\zeta} - z} \right)^{n} \\
&= \frac{(-1)^{n-1}}{(n-1)!} \left(\frac{\zeta - \bar{\zeta}}{\bar{\zeta} - z} \right)^{n} \frac{\overline{(\zeta - z)}^{n-1}}{\zeta - z}.
\end{aligned}
$$

Lemma 2 *For* $0 \leqslant \rho < n$, $z = \bar{z}$, $\partial_z^{\rho} G_n(z, \zeta) = 0$.

Proof If $z = \bar{z}$ and $0 \leqslant \rho < n$ then from (3)

$$
\begin{aligned}
&\partial_z^{\rho} G_n(z, \zeta) \\
&= \sum_{\nu=0}^{\rho-1} (-1)^{\nu} \binom{\rho}{\nu} \frac{(\zeta - z)^{(n-1-\rho)}}{(n-1-\nu)!} \frac{\overline{(\zeta - z)}^{n-1}}{(n-1)!} (\rho - \nu - 1)! \\
&\times \left(1 - \left(\frac{\zeta - z}{\bar{\zeta} - z} \right)^{\rho-\nu} \right) \\
&+ \frac{1}{(n-1)!^2} \sum_{\nu=1}^{\rho} \frac{1}{\nu} (\zeta - \bar{\zeta})^{\nu} (-1)^{\rho-\nu} \binom{\rho}{\rho-\nu} \frac{(n-1-\nu)!}{(n-1-\rho)!} \nu! \\
&\times (\zeta - z)^{n-1-\rho} \overline{(\zeta - z)}^{n-1-\nu} \\
&= \sum_{\nu=1}^{\rho} (-1)^{\rho-\nu} \binom{\rho}{\nu} \frac{(\nu-1)!}{(n-1-\rho+\nu)!} \frac{(\zeta - z)^{n-1-\rho} \overline{(\zeta - z)}^{n-1}}{(n-1)!} \\
&\times \left(1 - \left(\frac{\zeta - z}{\bar{\zeta} - z} \right)^{\nu} \right)
\end{aligned}
$$

$$+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{\rho}(-1)^{\rho-\nu}\binom{\rho}{\nu}(\nu-1)!\binom{n-1-\nu}{\rho-\nu}(\rho-\nu)!$$
$$\times(\zeta-z)^{n-1-\rho}\overline{(\zeta-z)}^{n-1-\nu}(\zeta-\bar{\zeta})^{\nu}$$
$$=\frac{1}{(n-1)!^2}\sum_{\nu=1}^{\rho}(-1)^{\rho-\nu}\binom{\rho}{\nu}(\nu-1)!\binom{n-1}{\rho-\nu}(\rho-\nu)!(\zeta-z)^{n-1-\rho}$$
$$\times\overline{(\zeta-z)}^{n-1}\sum_{\mu=1}^{\nu}\binom{\nu}{\mu}\left(\frac{\zeta-\bar{\zeta}}{\bar{\zeta}-z}\right)^{\mu}+\frac{1}{(n-1)!^2}\sum_{\nu=1}^{\rho}(-1)^{\rho-\nu}\binom{\rho}{\nu}$$
$$\times(\nu-1)!\binom{n-1-\nu}{\rho-\nu}(\rho-\nu)!(\zeta-z)^{n-1-\rho}\overline{(\zeta-z)}^{n-1-\nu}(\zeta-\bar{\zeta})^{\nu}$$
$$=\frac{1}{(n-1)!^2}\sum_{\mu=1}^{\rho}\sum_{\nu=\mu}^{\rho}(-1)^{\rho-\nu-1}\frac{\rho!(\nu-1)!}{(\nu-\mu)!\mu!}\binom{n-1}{\rho-\nu}$$
$$\times(\zeta-z)^{n-1-\rho}(\bar{\zeta}-z)^{n-1-\mu}(\zeta-\bar{\zeta})^{\mu}$$
$$+\frac{1}{(n-1)!^2}\sum_{\mu=1}^{\rho}(-1)^{\rho-\mu}\frac{\rho!}{\mu}\binom{n-1-\mu}{\rho-\mu}$$
$$(\zeta-z)^{n-1-\rho}\overline{(\zeta-z)}^{n-1-\mu}(\zeta-\bar{\zeta})^{\mu}$$

follows. From

$$\sum_{\nu=\mu}^{\rho}(-1)^{\nu-\mu}\binom{\nu-1}{\mu-1}\binom{n-1}{\rho-\nu}=\binom{n-1-\nu}{\rho-\mu}$$

follows

$$\partial_z^{\rho}G_n(z,\zeta)=-\frac{1}{(n-1)!^2}\sum_{\mu=1}^{\rho}(-1)^{\rho-1}\frac{\rho!}{\mu}\binom{n-1-\mu}{\rho-\mu}$$

$$\times(\zeta-z)^{n-1-\rho}(\bar{\zeta}-z)^{n-1-\mu}(\zeta-\bar{\zeta})^{\mu}+\frac{1}{(n-1)!^2}\sum_{\mu=1}^{\rho}(-1)^{\rho-\mu}$$

$$\times\frac{\rho!}{\mu}\binom{n-1-\mu}{\rho-\mu}(\zeta-z)^{n-1-\rho}(\bar{\zeta}-z)^{n-1-\nu}(\zeta-\bar{\zeta})^{\mu}=0.$$

Lemma 3 *For* $1\leqslant\rho<n$

$$(\partial_z\partial_{\bar{z}})^{\rho}G_n(z,\zeta)=G_{n-\rho}(z,\zeta)$$
$$+\sum_{\mu=0}^{\rho-1}\frac{(\zeta-\bar{\zeta})^{n-\mu-1}}{(n-\mu-2)!(n-\mu-1)!}(\partial_z\partial_{\bar{z}})^{\rho-\mu}\tilde{G}_{n-\mu}(z,\zeta),\qquad(6)$$

where $\tilde{G}_n(z,\zeta)$ *is some function satisfying*

$$\partial_z\partial_{\bar{z}}\tilde{G}_n(z,\zeta) = (z-\bar{z})^{n-2}\left(\frac{1}{\bar{\zeta}-z} - \frac{1}{\zeta-\bar{z}}\right) = (z-\bar{z})^{n-2}g_1(z,\zeta). \quad (7)$$

Proof If $\rho = 1$ then

$$\begin{aligned}
&(n-1)!^2\partial_z G_n(z,\zeta) = (-n-1)(\zeta-z)^{n-2}\overline{(\zeta-z)}^{n-1}\log\left|\frac{\bar{\zeta}-z}{\zeta-z}\right|^2\\
&-(n-1)\sum_{\nu=1}^{n-2}\frac{1}{\nu}(\zeta-z)^{n-2-\nu}(\zeta-\bar{z})^{n-1-\nu}(z-\bar{z})^{\nu}(\zeta-\bar{\zeta})^{\nu}\\
&-\frac{(z-\bar{z})^{n-1}(\zeta-\bar{\zeta})^{n-1}}{\bar{\zeta}-z},\\
&\partial_z\partial_{\bar{z}}G_n(z,\zeta) = (n-1)^2G_{n-1}(z,\zeta) + (n-1)(z-\bar{z})^{n-2}(\zeta-\bar{\zeta})^{n-1}\\
&\qquad\qquad\times\left(\frac{1}{\bar{\zeta}-z} - \frac{1}{\zeta-\bar{z}}\right) = (n-1)^2G_{n-1}(z,\zeta)\\
&\qquad\qquad+(n-1)(\zeta-\bar{\zeta})^{n-1}\partial_z\partial_{\bar{z}}\tilde{G}_n(z,\zeta).
\end{aligned}$$

By an induction argument from (6) follows

$$\begin{aligned}
&(\partial_z\partial_{\bar{z}})^{\rho+1}G_n(z,\zeta) = \partial_z\partial_{\bar{z}}G_{n-\rho}(z,\zeta)\\
&+\sum_{\mu=0}^{\rho-1}\frac{(\zeta-\bar{\zeta})^{n-\mu-1}}{(n-\mu-2)!(n-\nu-1)!}(\partial_z\partial_{\bar{z}})^{\rho-\mu+1}\\
&\times\tilde{G}_{n-\mu}(z,\zeta) = [(n-\rho-1)^2G_{n-\rho-1}(z,\zeta)\\
&\qquad\qquad+(n-\rho-1)(\zeta-\bar{\zeta})^{n-\rho-1}\partial_z\partial_{\bar{z}}\tilde{G}_{n-\rho}(z,\zeta)]\\
&+\sum_{\mu=0}^{\rho-1}\frac{(\zeta-\bar{\zeta})^{n-\mu-1}}{(n-\mu-2)!(n-\mu-1)!}(\partial_z\partial_{\bar{z}})^{\rho-\mu+1}\tilde{G}_{n-\mu}(z,\zeta)\\
&= G_{n-\rho-1}(z,\zeta) + \frac{(\zeta-\bar{\zeta})^{n-\rho-1}}{(n-\rho-2)!(n-\rho-1)!}\partial_z\partial_{\bar{z}}\tilde{G}_{n-\rho}(z,\zeta)\\
&+\sum_{\mu=0}^{\rho-1}\frac{(\zeta-\bar{\zeta})^{n-\mu-1}}{(n-\mu-2)!(n-\mu-1)!}(\partial_z\partial_{\bar{z}})^{\rho-\mu+1}\tilde{G}_{n-\mu}(z,\zeta)\\
&= G_{n-\rho-1}(z,\zeta)\\
&+\sum_{\mu=0}^{\rho}\frac{(\zeta-\bar{\zeta})^{n-\mu-1}}{(n-\mu-2)!(n-\mu-1)!}(\partial_z\partial_{\bar{z}})^{n-\mu+1}\tilde{G}_{n-\mu}(z,\zeta).
\end{aligned}$$

This is (6) for $\rho+1$ rather than for ρ.

Lemma 4 *For* $2 \leqslant 2\rho \leqslant n-2$

$$(\partial_z \partial_{\bar{z}})^{\rho} \tilde{G}_n(z,\zeta)$$
$$= (-1)^{\rho+1} \sum_{\mu=\rho+1}^{2\rho} \frac{(n-2)!}{(n-\mu)!}(z-\bar{z})^{n-\mu} \frac{(\rho-1)!}{(\mu-\rho-1)!} g_{1+2\rho-\mu}(z,\zeta), \quad (8)$$

where

$$g_{\alpha}(z,\zeta) = \frac{1}{(\bar{\zeta}-z)^{\alpha}} + \frac{(-1)^{\alpha}}{(\zeta-\bar{z})^{\alpha}} \quad (9)$$

for all $\alpha \in \mathbb{N}$.

Proof For $\rho = 1$ we have

$$\partial_z \partial_{\bar{z}} \tilde{G}_n(z,\zeta) = (z-\bar{z})^{n-2} g_1(z,\zeta),$$

then differentiating (8)

$$\partial_z (\partial_z \partial_{\bar{z}})^{\rho} \tilde{G}_n(z,\zeta) = (-1)^{\rho+1} \sum_{\mu=\rho+1}^{2\rho} [\frac{(n-2)!}{(n-\mu-1)!}(z-\bar{z})^{n-\mu-1}$$
$$\times \frac{(\rho-1)!}{(\mu-\rho-1)!} g_{1+2\rho-\mu}(z,\zeta) + \frac{(n-2)!}{(n-\mu)!}(1+2\rho-\mu)(z-\bar{z})^{n-\mu}$$
$$\times \frac{(\rho-1)!}{(\mu-\rho-1)} \frac{1}{(\bar{\zeta}-z)^{2+2\rho-\mu}}]$$

and

$$(\partial_z \partial_{\bar{z}})^{\rho+1} \tilde{G}_n(z,\zeta) = (-1)^{\rho} \sum_{\mu=\rho+1}^{2\rho} [\frac{(n-2)!}{(n-\mu-1)!}(z-\bar{z})^{n-\mu-2} \frac{(\rho-1)!}{(\mu-\rho-1)!}$$
$$\times g_{1+2\rho-\mu}(z,\zeta) + \frac{(n-2)!}{(n-\mu-1)!}(z-\bar{z})^{n-\mu-1} \frac{(\rho-1)!}{(\mu-\rho-1)!}$$
$$\times (1+2\rho-\mu) \frac{(-1)^{2\rho-\mu}}{(\zeta-\bar{z})^{2+2\rho-\mu}} + \frac{(n-2)!}{(n-\mu-1)!}(z-\bar{z})^{n-\mu-1}$$
$$\times \frac{(\rho-1)!}{(\mu-\rho-1)!}(1+2\rho-\mu) \frac{1}{(\bar{\zeta}-z)^{2+2\rho-\mu}}]$$
$$= (-1)^{\rho} \sum_{\mu=\rho+1}^{2\rho} [\frac{(n-2)!}{(n-\mu-2)!}(z-\bar{z})^{n-\mu-2} \frac{(\rho-1)!}{(\mu-\rho-1)!} g_{1+2\rho-\mu}(z,\zeta)$$

$$+\frac{(n-2)!}{(n-\mu-1)!}(z-\bar{z})^{n-\mu-1}\frac{(\rho-1)!}{(\mu-\rho-1)!}(1+2\rho-\mu)g_{2+2\rho-\mu}(z,\zeta)]$$

$$=(-1)^{\rho}\sum_{\mu=\rho+3}^{2\rho+2}[\frac{(n-2)!}{(n-\mu)!}(z-\bar{z})^{n-\mu}\frac{(\rho-1)!}{(\mu-\rho-3)!}g_{3+2\rho-\mu}(z,\zeta)$$

$$+(-1)^{\rho}\sum_{\mu=\rho+2}^{2\rho+1}\frac{(n-2)!}{(n-\mu)!}(z-\bar{z})^{n-\mu}\frac{(\rho-1)!}{(\mu-\rho-2)!}(2+2\rho-\mu)g_{3+2\rho-\mu}(z,\zeta)]$$

$$=(-1)^{\rho}\Bigg[\frac{(n-2)!}{(n-\rho-2)!}(z-\bar{z})^{n-\rho-2}(\rho-1)!\rho g_{\rho+1}(z,\zeta)$$

$$+\sum_{\mu=\rho+3}^{2\rho+1}\frac{(n-2)!}{(n-\mu)!}(z-\bar{z})^{n-\mu}\frac{(\rho-1)!}{(\mu-\rho-2)!}[\mu-\rho-2+2+2\rho-\mu]$$

$$\times g_{3+2\rho-\mu}(z,\zeta)+\frac{(n-2)!}{(n-2\rho-2)!}(z-\bar{z})^{n-2\rho-2}\frac{(\rho-1)!}{(\rho-1)!}g_1(z,\zeta)\Bigg]$$

$$=(-1)^{\rho}\Bigg[\frac{(n-2)!}{(n-\rho-2)!}(z-\bar{z})^{n-\rho-2}\frac{\rho!}{\mu!}g_{\rho+1}(z,\zeta)+\sum_{\mu=\rho+3}^{2\rho+1}\frac{(n-2)!}{(n-\mu)!}$$

$$\times(z-\bar{z})^{n-\mu}\frac{\rho!}{(\mu-\rho-2)!}g_{3+2\rho-\mu}(z,\zeta)$$

$$+\frac{(n-2)!}{(n-2\rho-2)!}(z-\bar{z})^{n-2\rho-2}\frac{\rho!}{\rho!}g_1(z,\zeta)\Bigg]$$

$$=(-1)^{\rho+2}\sum_{\mu=\rho+2}^{2\rho+2}\frac{(n-2)!}{(n-\mu)!}(z-\bar{z})^{n-\mu}\frac{\rho!}{(\mu-\rho-2)!}g_{3+2\rho-\mu}(z,\zeta)$$

follows. This reproduces (8) with $\rho+1$ replacing ρ.

Theorem 1 *For* $2\rho\le n-1$

$$\begin{aligned}&(\partial_z\partial_{\bar{z}})^{\rho}G_n(z,\zeta)=G_{n-\rho}(z,\zeta)\\&+\sum_{\nu=0}^{\rho-1}\sum_{\mu=0}^{\rho-1-\nu}(-1)^{\rho-\mu-1}\frac{(\rho-\mu-1)!}{(n-\nu-\rho-1)!(n-1-\mu)!\nu!}\\&\times(\zeta-\bar{\zeta})^{n-\mu-1}(z-\bar{z})^{n-\rho-1-\nu}g_{\rho-\mu-\nu}(z,\zeta).\end{aligned}\tag{10}$$

Remark 1 $(\partial_z\partial_{\bar z})^\rho G_n(z,\zeta)=0$ on $z=\bar z$ for $2\rho<n$ i.e. $0<\rho\leqslant[\frac{n-1}{2}]$ because $0<n-\rho-1-\nu$ i.e. $\nu<n-\rho-1$, as $\nu\leqslant\rho-1<n-\rho-1$.

Proof From (6) and (8)

$$(\partial_z\partial_{\bar z})^\rho G_n(z,\zeta)-G_{n-\rho}(z,\zeta)=\sum_{\mu=0}^{\rho-1}\frac{(\zeta-\bar\zeta)^{n-\mu-1}}{(n-\mu-2)!(n-\mu-1)!}(-1)^{\rho-\mu-1}$$

$$\times\sum_{\nu=\rho-\mu+1}^{2(\rho-\mu)}\frac{(n-\mu-2)!}{(n-\mu-\nu)!}(z-\bar z)^{n-\mu-\nu}\frac{(\rho-\mu-1)!}{(\nu-\rho+\mu-1)!}g_{1+2\rho-2\mu-\nu}(z,\zeta)$$

$$=\sum_{\mu=0}^{\rho-1}\sum_{\nu=0}^{\rho-1-\mu}(-1)^{\rho-1-\mu}\frac{(\rho-\mu-1)!}{(n-\mu-1)!(n-\rho-\nu-1)!\nu!}$$

$$\times(\zeta-\bar\zeta)^{n-\mu-1}(z-\bar z)^{n-\rho-\nu-1}g_{\rho-\mu-\nu}(z,\zeta)$$

$$=\sum_{\mu=0}^{\rho-1}\sum_{\nu=\mu}^{\rho-1}(-1)^{\rho-1-\mu}\frac{(\rho-\mu-1)!}{(n-1-\mu)!(n-\rho-1-\nu+\mu)!(\nu-\mu)!}$$

$$\times(\zeta-\bar\zeta)^{n-\mu-1}(z-\bar z)^{n-\rho-\nu+\mu-1}g_{\rho-\nu}(z,\zeta)$$

$$=\sum_{\nu=0}^{\rho-1}\sum_{\mu=0}^{\nu}(-1)^{\rho-1+\mu-\nu}\frac{(\rho-\nu+\mu-1)!}{(n-1-\nu+\mu)!(n-\rho-1-\mu)!\mu!}$$

$$\times(\zeta-\bar\zeta)^{n-\nu+\mu-1}(z-\bar z)^{n-\rho-1-\mu}g_{\rho-\nu}(z,\zeta).$$

Lemma 5 *For* $0\leqslant\sigma,\rho,\ \sigma+2\rho\leqslant n-1$

$$\partial_z^\sigma(\partial_z\partial_{\bar z})^\rho G_n(z,\zeta)=\frac{1}{(n-1-\rho)!^2}\partial_z^\sigma G_{n-\rho}(z,\zeta)$$

$$+\sum_{\nu=0}^{\rho-1}\sum_{\mu=0}^{\rho-1-\nu}(-1)^{\rho-\mu-1}\frac{(\rho-\mu-1)!(\zeta-\bar\zeta)^{n-\mu-1}}{(n-\rho-1-\nu)!(n-1-\mu)!\nu!}$$

$$\times\left[\frac{(n-\rho-1-\nu)!}{(n-\rho-1-\nu-\sigma)!}(z-\bar z)^{n-\rho-1-\nu-\sigma}g_{\rho-\mu-1}(z,\zeta)\right.$$

$$+\sum_{\lambda=0}^{min\{\sigma-1,n-\rho-1-\nu\}}\frac{(n-\rho-1-\nu)!}{(n-\rho-1-\nu-\lambda)!}\frac{(\rho-\mu-\nu+\sigma-\lambda-1)!}{(\rho-\mu-\nu-1)!}$$

$$\left.\times(z-\bar z)^{n-\rho-1-\nu-\lambda}\frac{1}{(\bar\zeta-z)^{\rho-\mu-\nu+\sigma-\lambda}}\right].\qquad(11)$$

Proof Formula (11) follows from (10) by differentiation.

Corollary 2 *If* $\sigma + 2\rho \leqslant n-1$, *then*

$$\partial_z^\sigma (\partial_z \partial_{\bar{z}})^\rho G_n(z,\zeta) = 0$$

for $z = \bar{z}$.

Proof Because $\sigma \leqslant n-\rho-1$ then $\partial_z^\sigma G_{n-\rho}(z,\zeta) = 0$ for $z = \bar{z}$. From $\sigma + 2\rho \leqslant n-1$ if follows $\sigma < n - 2\rho \leqslant n-\rho-1-\nu$ for $0 \leqslant \nu \leqslant \rho - 1$ i.e. $0 < n-\rho-1-\nu-\lambda$ for $0 \leqslant \lambda \leqslant \sigma < n-\rho-1-\nu$. Hence setting $z = \bar{z}$ in (11) and applying Lemma 2 the result follows.
Remark 2 The result of Corollary 2 can be reformulated: if $\mu + \nu \leqslant n-1$, then

$$\partial_z^\mu \partial_{\bar{z}}^\nu G_n(z,\zeta) = 0$$

for $z = \bar{z}$.

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Preface A

Part A contains a selection of papers partly associated with the special session on wavelets, other multiscale methods and their applications at the 995th AMS Meeting in Athens, Ohio, March 26-27, 2004, and the special session on the multiscale methods and sampling in time-frequency analysis at the 1000th AMS Meeting in Albuquerque, New Mexico, October 16-17, 2004.

The papers cover topics such as path-connectivity of tight frame wavelets, blockwise–shrinkage wavelet estimates, biorthogonal spline wavelets, and framebased nonuniform sampling.

There are many people who provided valuable assistance during the various stages of this issue. We would like to thank our guest editors Joe Lakey of the New Mexico State University, Las Cruces, En-Bing Lin of the University of Toledo, and Xiaoping Shen of the Ohio University, Athens, for their fantastic editing work. Thanks are also due to George Anastassiou, the Editor-in-Chief, and Tian-Xiao He, the Associate Editor of the Journal of Applied Functional Analysis. Most of all, we are grateful to all of contributors to this issue.

Editors of the Special Issue: Joe Lakey, En-Bing Lin, Xiaoping Shen, and Tian-Xiao He

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The Path-Connectivity of s-Elementary Tight Frame Wavelets

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Abstract

An s-elementary tight frame wavelet is a tight frame wavelet whose Fourier transform is of the form $\frac{1}{\sqrt{2\pi}}\chi_E$ for some measurable set $E \subset \mathbb{R}$. It is known that the frame bound of an s-elementary tight frame wavelet is an integer. In this paper, we prove that for any integer $k \geq 1$, the set of all s-elementary tight frame wavelets of frame bound k is a path-connected component of the set of all s-elementary tight frame wavelets (which is itself a non path-connected set).

1 Introduction

The topological property of various families of wavelets is an interesting topic in the study of wavelet theory. The question concerning the path-connectedness of the set of all orthonormal wavelets was first raised in [7]. Similar questions were raised and studied in [5, 8, 9, 10, 11] about the set of all MRA-wavelets, tight frame wavelets, MRA tight frame wavelets and s-elementary frame wavelets. In [8, 11], it is shown that the set of MRA-wavelets is path-connected. In [10], it is shown that the set of s-elementary

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wavelets are path-connected. The proofs of these theorems were based on the complete characterizations of the MRA-wavelets and s-elementary wavelets. While the complete characterization of the s-elementary frame wavelets is still an open question, it has been shown that the set of s-elementary frame wavelets is path-connected as well [5]. In this paper, we will prove the path-connectedness of the s-elementary tight frame wavelets (with the same frame bounds).

A set $S \subset L^2(\mathbb{R})$ is said to be *path-connected* under the norm topology of $L^2(\mathbb{R})$ if for any two members $f, g \in S$, there exists a mapping $\gamma : [0,1] \longrightarrow S$ such that the function $\gamma(t)$ is continuous in the norm of $L^2(\mathbb{R})$ and $\gamma(0) = f$, $\gamma(1) = g$.

Let T and D be the translation and dilation unitary operators acting on $\mathcal{H}$ defined by

$$(Tf)(t) = f(t-1) \text{ and } \quad (Df)(t) = \sqrt{2}f(2t), \forall f \in L^2(\mathbb{R})$$

and let $\mathcal{F}$ be the (unitary) Fourier-Plancherel transform defined by

$$(\mathcal{F}\{)() = \frac{}{\sqrt{\pi}} \int_{\mathbb{R}} {}^{-}\{(), \forall \{ \in \pounds(\mathbb{R}) \cap \mathbb{L}^{\nVdash}(\mathbb{R}).$$

$\mathcal{F}(\{)$ is sometimes written as $\widehat{f}$ as well.

An *orthogonal wavelet* of $L^2(\mathbb{R})$ is a function $\psi(t)$ in $L^2(\mathbb{R})$ with unit norm such that $\{2^{\frac{n}{2}}\psi(2^n t - \ell) : n, \ell \in \mathbb{Z}\} = \{\mathbb{D}^{\kappa}\mathbb{T}^{\ell}\psi : \kappa, \ell \in \mathbb{Z}\}$ constitutes an orthonormal basis for $L^2(\mathbb{R})$. Let $\mathcal{W}$ be the set of all orthonormal wavelets of $L^2(\mathbb{R})$. The set of the well known MRA-wavelets is a subset of $\mathcal{W}$. It is shown that the set of MRA-wavelets is path-connected [8, 11]. However, the path-connectedness of $\mathcal{W}$ remains an open question at this time.

2 The s-elementary Frame Wavelets

A function $\psi \in L^2(\mathbb{R})$ is called a *frame wavelet* for $L^2(\mathbb{R})$ if there exist two positive constants $0 < a \leq b$ such that for any $f \in L^2(\mathbb{R})$,

$$a\|f\|^2 \leq \sum_{n,\ell \in \mathbb{Z}} |\langle f, D^n T^{\ell} \psi \rangle|^2 \leq b\|f\|^2. \tag{2.1}$$

A number a with this property is called a *lower frame bound* of ψ and a number b with this property is called an *upper frame bound* of ψ. Let a_0 be the supremum of all such numbers a and let b_0 be the infimum of of all such numbers b. If $a_0 = b_0$, then ψ is called a *tight frame wavelet* and the number $a_0 = b_0$ is called the *tight frame bound* of ψ. Furthermore, if $a_0 = b_0 = 1$, then ψ is called a *normalized tight frame wavelet*. Let E be a Lebesgue measurable set of finite measure and χ_E be its corresponding characteristic function. If the function $\psi_E \in L^2(\mathbb{R})$ defined by $\widehat{\psi_E} = \frac{1}{\sqrt{2\pi}}\chi_E$ is a frame wavelet, a tight frame wavelet or a normalized tight frame wavelet for $L^2(\mathbb{R})$, then the set E is called a *frame wavelet set*, a *tight frame wavelet set* or a *normalized tight frame wavelet set* for $L^2(\mathbb{R})$ respectively. The corresponding function ψ_E is called an *s-elementary*, an *s-elementary tight* or an *s-elementary normalized tight* frame wavelet. Let us denote the set of all s-elementary tight frame wavelets with tight frame bound k by $\mathcal{W}_{\{}()$. By a result from [3], the tight frame bound of an s-elementary tight frame wavelet must be an integer, i.e., $k \in \mathbb{N}$ for each $\mathcal{W}_{\{}()$. In particular, $\mathcal{W}_{\{}()$ is the set of all s-elementary normalized tight frame wavelets. It is shown in [6] that $\mathcal{W}_{\{}()$ is path-connected. In this paper, we will show that in general, for any $k \geq 2$, the set $\mathcal{W}_{\{}()$ is also path-connected. We will need the following concepts and results for this purpose. The details can be found in [3].

Let E be a measurable set. $x, y \in E$ are δ-equivalent if $x = 2^n y$ for some integer n. The δ-index of a point x in E is the number of elements in its δ-equivalent class and is denoted by $\delta_E(x)$. Let $E(\delta, k) = \{x \in E : \ \delta_E(x) = k\}$, then E is the disjoint union of the sets $E(\delta, k)$. Furthermore, each $E(\delta, k)$ $(k \geq 1)$ is Lebesgue measurable and is a disjoint union of k measurable sets $\{E^j(\delta, k)\}$, $1 \leq j \leq k$, such that $E^j(\delta, k) \overset{\delta}{\sim} E^{j'}(\delta, k)$ for any $1 \leq j, \ j' \leq k$. If we let $\Delta(E) = \cup_{n\in\mathbb{Z}} 2^n E$, then we have $\Delta(E) = \Delta(\cup_{k\geq 1} E^1(\delta, k))$. A set E is called a *2-dilation generator* of $\mathbb{R}$ if $E = E(\delta, 1)$ and $\Delta(E) = \mathbb{R}$.

In the case of translation, we say that $x, y \in E$ are 2π-translation equivalent if $x = y + 2n\pi$ for some integer n. The τ-index of a point x in E is the number of elements in its 2π-translation equivalent class and is denoted by $\tau_E(x)$. Let $E(\tau, k) = \{x \in E : \ \tau_E(x) = k\}$. Then E is the disjoint union of the (measurable) sets $E(\tau, k)$. Define $\tau(E) = \cup_{n\in\mathbb{Z}}(E \cap ([2n\pi, 2(n + 1)\pi) - 2n\pi))$. This is a disjoint union if and only if $E = E(\tau, 1)$. Finally, each $E(\tau, k)$ is a disjoint union of k measurable sets $\{E^{(j)}(\tau, k)\}$, $1 \leq j \leq k$,

such that $E^{(j)}(\tau,k) \overset{\tau}{\sim} E^{j'}(\tau,k)$ for any $1 \le j,\ j' \le k$.

The decompositions of $E(\delta,k)$ (resp. $E(\tau,k)$) into $E^j(\delta,k)$ (resp. $E^{(j)}(\tau,k)$) are not unique in general. However, one of them is guaranteed by the procedure of construction in [4] and the set $E(\tau,1)$ is uniquely determined by E. However in this paper we will not be dealing with the decompositions hence there shall be no confusion.

The following theorem is quoted from [3], which characterizes the tight frame wavelet sets (hence the s-elementary tight frame wavelets).

Theorem 2.1 *Let E be a Lebesgue measurable set with finite measure. Then E is a tight frame wavelet set if and only if $E = E(\tau,1) = E(\delta,k)$ for some $k \ge 1$ and $\Delta(E) = \mathbb{R}$. In particular, the frame bound of an s-elementary tight frame wavelet is a positive integer.*

3 The Path-Connectivity of Tight s-elementary Frame Wavelets

Let us recall that the path-connectedness of the set $\mathcal{W}_{\{\}}()$ means that for any two tight frame wavelet sets E, F and their corresponding tight frame wavelets ψ_E, ψ_F, there exists a continuous mapping $\gamma:\ [0,1] \longrightarrow \mathcal{W}_{\{\}}()$ such that $\gamma(t) \in \mathcal{W}_{\{\}}()$ for each t and $\gamma(0) = \psi_E$, $\gamma(1) = \psi_F$. We say that $\{E_t\}$ is a path connecting E and F if χ_{E_t} is a continuous (in the $L^2(\mathbb{R})$ norm) path connecting χ_E and χ_F. The following lemma is immediate and its proof is left to the reader.

Lemma 3.1 *Let ψ_E and $\psi_F \in \mathcal{W}_{\{\}}()$ be two s-elementary tight frame wavelets of frame bound k with E, F being their corresponding tight frame wavelet sets. Then the following statements are equivalent:*

(A) ψ_E and ψ_F are connected by a path in $\mathcal{W}_{\{\}}()$ that is continuous in the $L^2(\mathbb{R})$ norm;

(B) There exists a path $\{E_t\}$ connecting E and F;

(C) For each $t \in [0,1]$, there exists a measurable set E_t such that: (1) E_t is a tight frame set of frame bound k for each t; (2) $E_0 = E$ and $E_1 = F$;

(3) E_t is continuous in t, that is, for any $t_0 \in [0,1]$ and any $\epsilon > 0$, there exists a $\epsilon_1 > 0$ such that for any $t \in [0,1] \cap (t_0 - \epsilon_1, t_0 + \epsilon_1)$, we have $\mu(E_t \setminus E_{t_0}) + \mu(E_{t_0} \setminus E_t) < \epsilon$, where μ is the Lebesgue measure.

Lemma 3.2 *Let E be a tight frame wavelet set of frame bound k and $c \in (0,\pi)$. Then for any $\epsilon > 0$, there exists $\epsilon_1 > 0$ such that for any measurable set $A \subset [-\pi,-c) \cup [c,\pi]$, if $\mu(A) < \epsilon_1$, then $\mu(E \cap \Delta(A)) < \epsilon$.*

Proof. Since $E = E(\tau, 1)$, it follows that $\mu(E) \leq 2\pi$. Hence for any $\epsilon > 0$, there exists $M > 0$ such that $\mu(E \cap [M,\infty)) + \mu(E \cap (-\infty,-M]) < \frac{\epsilon}{2}$. On the other hand, it is clear that there exists a constant $N > 0$ such that $\mu(\Delta(A) \cap [-M,M]) \leq N \cdot \mu(A)$. Thus if $\mu(A) < \epsilon_1 = \frac{\epsilon}{2N}$, then we have $\mu(\Delta(E)) < \epsilon$.

From Lemma 3.2 we immediately have the following

Corollary 3.3 *Let E be a tight frame wavelet set of frame bound k and $c > 0$ be an arbitrary given positive number. If $F_t \subset [-\pi,-c) \cup [c,\pi]$ is continuous in t, then so is $E \cap \Delta(F_t)$.*

We are now ready to state and prove our main theorem.

Theorem 3.4 *The set $\mathcal{W}_{\{}()$ is path-connected.*

Proof. By Lemma 3.1, it suffices to show that any s-elementary tight frame wavelet set E of frame bound k is connected to the tight frame wavelet set $I = [-\pi, -\frac{\pi}{2^k}] \cup [\frac{\pi}{2^k}, \pi]$. For each $t \in [0,1]$, define

$$I_t = [-\pi, -(1-\frac{t}{2})\pi) \cup [(1-\frac{t}{2})\pi, \pi), \tag{3.1}$$

$$J_t^1 = I \cap \Delta(I_t) = \cup_{-(k-1)\leq n\leq 0} 2^n I_t, \tag{3.2}$$

$$F_t^1 = E \cap \Delta(I_t) = E \cap \Delta(J_t^1). \tag{3.3}$$

Observe that the set $G_t = J_t^1 \cup (E \setminus F_t^1)$ satisfies the condition $G_t = G_t(\delta, k)$. However, G_t may not be an s-elementary tight frame wavelet set since $G_t = G_t(\tau, 1)$ may not hold. In other word, we cannot use G_t as the path we need yet. We will now make modifications to G_t in order to obtain an s-elementary tight frame wavelet set.

For any measurable set A, define $\mathfrak{T}(\mathcal{A}) = \cup_{\setminus \in \mathbb{Z}}(\mathcal{A} + \setminus \pi)$ and let $H_t^1 = (E \setminus F_t^1) \cap \mathfrak{T}(\mathcal{J})$. Observe that $J_t^1 \cap (E \setminus F_t^1) = \emptyset$ since $J_t^1 \cap E \subset F_t^1$. It follows that $H_t^1 \subset (-\infty, -\pi] \cup [\pi, \infty)$. We now define $J_t^2 = I \cap \Delta(H_t^1)$ This process can now be repeated:

$$\begin{aligned} F_t^2 &= E \cap \Delta(J_t^2), \\ H_t^2 &= (E \setminus F_t^2) \cap \mathfrak{T}(\mathcal{J}), \\ J_t^3 &= I \cap \Delta(H_t^2), \\ F_t^3 &= E \cap \Delta(J_t^3), \\ H_t^3 &= (E \setminus F_t^3) \cap \mathfrak{T}(\mathcal{J}), \\ J_t^4 &= I \cap \Delta(H_t^3), \\ .. &\quad ... \end{aligned}$$

We will leave it to our reader to verify that:

$$J_t^i \cap J_t^j = \emptyset \text{ whenever i} \neq \text{j}; \tag{3.4}$$

and

$$\mu(J_t^n) \leq \frac{1}{2}\mu(H_t^{n-1}) \leq \frac{1}{2}\mu(J_t^{n-1}). \tag{3.5}$$

So in general we have

$$\mu(J_t^n) \leq \frac{1}{2^{n-1}}\mu(J_t^1). \tag{3.6}$$

Finally we let

$$J_t = \cup_{n \geq 1} J_t^n, \tag{3.7}$$

$$F_t = E \cap \Delta(J_t) \tag{3.8}$$

$$E_t = J_t \cup (E \setminus F_t). \tag{3.9}$$

From this definition, it is immediate that $E_0 = E$ and $E_1 = I$. By Lemma 3.1, it suffices to show that E_t is a tight frame wavelet set for each t and is continuous in t.

For any $t_2 > t_1$, it is clear that $J_{t_1} \subset J_{t_2}$ so $J_{t_1} \setminus J_{t_2} = \emptyset$. On the other hand, we have $J_{t_2} \setminus J_{t_1} = J'_{t_1,t_2}$ where J'_{t_1,t_2} is similarly defined as J_t by replacing I_t with $I'_{t_1,t_2} = [-(1-\frac{t_1}{2})\pi, -(1-\frac{t_2}{2})\pi) \cup [(1-\frac{t_2}{2})\pi, (1-\frac{t_1}{2})\pi)$. Using the same argument to obtain the inequality (3.6) above, we have

$$\begin{aligned} \mu(J_{t_2} \setminus J_{t_1}) &= \mu(J'_{t_1,t_2}) \\ &\leq 4\mu(I'_{t_1,t_2}) \\ &= 4(t_2 - t_1)\pi. \end{aligned}$$

It follows from this that J_t is continuous. By Corollary 3.3, F_t is continuous hence E_t is continuous.

Firstly, to see that $E_t = E(\tau, 1)$, assume that it is not true. Then we must have some $x \in J_t$ such that $x + 2n\pi \in E \setminus F_t$ for some non-zero $n \in \mathbb{Z}$, since J_t and $E \setminus F_t$ are both 2π-translation redundancy free sets. By the definition of J_t, we have $x \in J_t^i$ for some $i \geq 1$ so $x + 2n\pi \in \mathcal{T}(\mathcal{J})$. On the other hand, since $x + 2n\pi \notin F_t = E \cap \Delta(J_t)$, $x + 2n\pi \notin F_t^i = E \cap \Delta(J_t^i) \subset E \cap \Delta(J_t)$. It follows that $x + 2n\pi \in H_t^i = (E \setminus F_t^i) \cap \mathcal{T}(\mathcal{J})$, hence $x + 2n\pi \in \Delta(J_t^{i+1})$ by the definition of J_t^{i+1}. But this contradicts the fact that $x + 2n\pi \notin F_t = E \cap \Delta(J_t)$. This proves that $E_t = E(\tau, 1)$.

Secondly, we need to show that $E_t = E(\delta, k)$, that is, each point in E_t must have δ-index k. Let $x \in E_t$. There are two situations: $x \in J_t$ or $x \in E \setminus F_t$ (notice that J_t and $E \setminus F_t$ are disjoint sets).

Case 1. $x \in J_t$. Thus $x \in J_t^i$ for some i. Since $J_t^i = I \cap \Delta(H_t^{i-1})$, it clearly contains all of the k points in I that are δ-equivalent to x (including x itself), thus the δ-index of x in E_t is at least k. Furthermore, J_t is a subset of I, so the δ-index of x in E_t is at most k.

Case 2. $x \in E \setminus F_t$. Since $x \notin \Delta(J_t)$, none of its other $k-1$ δ-equivalent points (that are in E) is in $\Delta(J_t)$ either. Thus $\delta_{E_t}(x) = k$ as well.

Finally, we need to show that $\Delta(E_t) = \mathbb{R}$ (modulus a zero measure set). Since $\Delta(E) = \mathbb{R}$, for any $y \in \mathbb{R}$, we have $x \in E$ such that $2^p x = y$ for some $p \in \mathbb{Z}$. If $x \notin \Delta(J_t)$, then $x \in E_t$ and we are done. If $x \in \Delta(J_t)$, then $2^q x \in J_t$ for some $q \in \mathbb{Z}$ and $y = 2^{p-q}(2^q x) \in \Delta(J_t) \subset \Delta(E_t)$. This finishes our proof of the main theorem.

We will end this paper with the following remark.

Remark 3.5 *For any $k_1 \neq k_2 \in \mathbb{N}$, the set $\mathcal{W}_{\{}() \cup \mathcal{W}_{\{}()$ is not path-connected. In fact, for any $\psi_{E_1} \in \mathcal{W}_{\{}()$ and $\psi_{E_2} \in \mathcal{W}_{\{}()$, there is no path in $\mathcal{W}_{\{}() \cup \mathcal{W}_{\{}()$ connecting them. We leave this to our reader as an exercise.*

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UNIVERSAL LOWER BOUNDS FOR BLOCKWISE–SHRINKAGE WAVELET ESTIMATION OF A SPIKE

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Abstract

A spike function plays an important role in the theoretical analysis and numerical study of adaptive wavelet regression estimates. This article uses this function to develop nonasymptotic lower bounds for the mean squared error of popular blockwise–shrinkage wavelet estimates including Efromovich–Pinsker, Stein and James–Stein blockwise–shrinkage adaptive estimators. The new feature of the lower bounds is that a single spike function is used to analyze vast classes of blockwise–shrinkage estimates while before a spike function depended on a studied estimator. As a result, the obtained lower bounds allow one to deduce minimax lower bounds and describe classes of estimates that cannot attain optimal rates of the mean squared error convergence. The developed nonasymptotic lower–bound methodology bridges minimax and numerical approaches traditionally used for the analysis of wavelet estimates.

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Keywords and Phrases: Adaptation; Non-asymptotic; Regression; Stein shrinkage.

1 Introduction

Asymptotic minimax lower bounds for the mean squared error of wavelet regression estimates have become an important statistical tool in assessing optimality of adaptive wavelet estimates; see a discussion in [2,9,11,14,15]. In this article a special class of adaptive wavelet estimates — blockwise–shrinkage wavelet estimates — will be considered. These estimates have become popular in theoretical and applied statistics because they are asymptotically sharp minimax, superefficient, have nice plug-in properties for estimation of both linear and nonlinear functionals, robust toward distribution of regression errors and the design of predictors and perform well for small sample sizes; see a discussion in [1–6,8,13.15].

A traditional asymptotic minimax study of wavelet estimates is based on exploring a global minimax lower bound where the maximum is taken over a wide class of possible estimated functions (Hölder function class is a popular choice in the literature) and the minimum is taken over all possible estimates. Such a study may imply minimax rates of convergence over a considered function class and then allow the statistician to define an estimate as a minimax one if its risk attains the lower bound; see a discussion in [6,8,14]. Recently, it has been suggested in [9,10] to complement known asymptotic lower minimax bounds by nonasymptotic bounds calculated for a given estimated function and a class of blockwise–shrinkage estimates that are known to be minimax in the traditional sense. Following [6,11], two main types of estimated functions are considered in those references. The former is a smooth background where all major results can be established for a no–signal case; see a discussion in [9]. The latter is a spike function which is considered in this article so let us comment on why it is of a special interest. First of all, this function plays a pivotal role in applications because non–zero wavelet coefficients are sparse and thus a reliable recovery of spikes is paramount for an adaptive wavelet estimation; see [6,14]. Second, spike is a traditional test–function used in numerical studies; see a discussion in [8,14]. As a result, if a wavelet estimate does not perform well on a spike function then the statistician should raise a question about its feasibility. Third, a spike function

has been always used in the asymptotic minimax literature to establish minimax lower bounds for mean squared errors of different statistical estimates including kernel, spline and orthogonal series ones. The interested reader can find an interesting discussion in [8, ch.7].

First known nonasymptotic lower bounds for estimation of spike functions by blockwise–shrinkage estimates were suggested in [10]. Let us recall that there exists a vast class of these estimates, in terms of different portfolios of blocks, thresholds and shrinkage procedures, that are sharp minimax under the Mean Integrated Squared Error (MISE) criteria as well as under the Mean Squared Error (MSE) criteria for no–signal estimated function; see [3,6,9,15]. On the other hand, a minimax approach under the MSE criteria allows the statistician to dramatically shrink a class of admissible wavelet estimates; see a discussion in [2]. The lower–bound methodology, suggested in [10] for the case of an estimated spike function, has shed a new light on the issue and it also developed a set of necessary conditions for a blockwise–shrinkage estimate to attain desired rates of the MISE convergence. On the other hand, main deficiency of those results is the utilization of spike functions specifically designed for each studied estimate. As a result, it has been impossible to suggest a universal lower bound, based on a single estimated function, which could imply a corresponding nonasymptotic minimax lower bound.

It is definitely desirable to develop nonasymptotic lower bounds that allow the statistician to analyze a whole class of wavelet estimates via a single underlying test–function; note that this is the approach used by numerical methods. Theoretical results of this article allow the statistician to do this and they justify and shed a new light on the theory and methods developed in [10]. Let us also note that, among known publications in the wavelet literature, the methodology of [12] (where smart numerical methods are used to calculate exact risks of wavelet estimates) is probably the closest one to the considered methodology.

The context of the article is as follows. The setting and studied estimates are explained in Section 2. Lower bounds and corollaries are presented in Section 3, and their discussion

is placed in Section 4. Section 5 contains proofs.

2 Setting and wavelet estimates

Consider a classical homoscedastic regression model $Y_l = f(l/n) + v^{1/2}\epsilon_l, l = 1, 2, \ldots, n$, where f is an unknown bounded regression function, $\epsilon_1, \ldots, \epsilon_n$ are zero mean and unit variance random variables, and v is the known variance of regression errors. Under mild assumptions, this model is equivalent to a series model written in a wavelet basis domain. The wavelet basis is assumed to be periodized and orthonormal on $[0, 1]$, and it is generated by a pair of compactly supported scaling function and wavelet function, the latter will be denoted as ψ. For a jth resolution scale set $\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k)$, $x \in [0, 1]$, and assume that the primary resolution level j_0 is large enough to ensure that the supports of the scale and wavelet functions at level j_0 are not the whole of the unit interval. Because we are exploring lower bounds, we can always restrict our attention to regression functions that are expanded into a wavelet series

$$g(x) = \sum_{j=j_0}^{\infty} \sum_{k=1}^{2^j} \theta_{j,k}\psi_{j,k}(x), \quad x \in [0, 1], \tag{1}$$

where $\theta_{j,k} = \int_0^1 g(x)\psi_{j,k}(x)dx$ are wavelet coefficients. The introduced function g has no scaling component (all its scaling coefficients are zero), and as we shall see only finer resolution scales will be used for obtaining wished lower bounds.

The statistical assumption is a traditional one: the sample size n is dyadic and for all $j_0 \le j \le \log_2(n/\ln(n))$ and all k considered there exist statistics $\tilde{\theta}_{j,k}$ (empirical wavelet coefficients) such that $\tilde{\theta}_{j,k} = \theta_{j,k} + v^{1/2}n^{-1/2}\xi_{j,k}$ where $\{\xi_{j,k}\}$ are independent, zero mean and unit variance random variables (that are typically assumed to be normal but this assumption is not necessary in the considered lower–bound analysis). The interested reader can find more about wavelet bases and the assumptions in [8,11,14].

This article studies blockwise shrinkage estimates. These estimates employ a special shrinkage factor for empirical wavelet coefficients. The factor mimics a benchmark factor

used by an oracle that knows some functionals of underlying wavelet coefficients. Thus, let us begin with the oracle's shrinkage factor. Consider a particular wavelet coefficient $\theta_{j,k}$ and assume that it is estimated using a block $\{\tilde\theta_{j,t},\ t \in B_{j,m}\}$ of empirical wavelet coefficients. In what follows it is always assumed that $k \in B_{j,m}$, each wavelet coefficient (index) belongs to a single block and only consecutive coefficients from the same scale are included into a block. Let us denote the cardinality (size, length) of a block $B_{j,m}$ by $L_{j,m}$. Then, according to [6,8], the benchmark blockwise–shrinkage oracle is

$$\hat\theta^*_{j,k} = \frac{L^{-1}_{j,m}\sum_{t\in B_{j,m}}\theta^2_{j,t}}{L^{-1}_{j,m}\sum_{t\in B_{j,m}}\theta^2_{j,t} + vn^{-1}}\tilde\theta_{j,k}. \tag{2}$$

Two statistical procedures, mimicking the oracle, are suggested in the literature. The first one is

$$\hat\theta_{j,k} = \frac{L^{-1}_{j,m}\sum_{t\in B_{j,m}}\tilde\theta^2_{j,t} - vn^{-1}}{L^{-1}_{j,m}\sum_{t\in B_{j,m}}\tilde\theta^2_{j,t}} I\Big(L^{-1}_{j,m}\sum_{t\in B_{j,m}}\tilde\theta^2_{j,t} > (1+\lambda^2_{j,m})vn^{-1}\Big)\tilde\theta_{j,k}, \tag{3}$$

and it is referred to as Efromovich–Pinsker (EP) blockwise shrinkage. Here $\{\lambda^2_{j,m}\}$ are positive parameters that are called thresholds (some authors refer to $1+\lambda^2_{j,m}$ as the penalty). Note that EP procedure combines a naive mimicking of (2) with the idea of hard thresholding; a comprehensive discussion of the procedure can be found in [7,8]. The second blockwise shrinkage procedure combines a naive mimicking of the oracle (2) with the idea of a soft thresholding,

$$\bar\theta_{j,k} = \frac{L^{-1}_{j,m}\sum_{t\in B_{j,m}}\tilde\theta^2_{j,t} - (1+\mu_{j,m})vn^{-1}}{L^{-1}_{j,m}\sum_{t\in B_{j,m}}\tilde\theta^2_{j,t}} I(L^{-1}_{j,m}\sum_{t\in B_{j,m}}\tilde\theta^2_{j,t} > (1+\mu_{j,m})vn^{-1})\tilde\theta_{j,k}. \tag{4}$$

Here parameters $\mu_{j,m}$, that are not necessarily positive, also referred to as thresholds. In particular, if $\mu_{j,m} = -2/L_{j,m}$, $L_{j,m} > 2$ then the shrinkage (4) is called the (positive–part) James-Stein one, and the shrinkage with $\mu_{j,m} \ge 0$, $L_{j,m} > 4$ is called the Stein shrinkage. This type of shrinkage is very popular in the classical multiparameter estimation literature, and the interested reader can find a discussion of the soft–thresholding procedures in [2,3,6,11,15].

Now we are in a position to define estimates of the regression function. A corresponding to the shrinkage (2) oracle for estimation of the regression function (the benchmark pseudo–estimate) is

$$\hat{f}^*(x) = \sum_{j=j_0}^{\log_2(n/\ln(n))} \sum_m \sum_{k\in B_{j,m}} \hat{\theta}^*_{j,k}\psi_{j,k}(x), \quad x \in [0,1]. \tag{5}$$

Here and in what follows the summation over m is taken over all blocks on jth scale. Let us also note that the cutoff $\log_2(n/\ln(n))$ is a traditional one in the literature because it is sufficient for minimax estimation of regression functions from all studied in the literature function classes; see [6,14].

A corresponding to the shrinkage (3) Efromovich–Pinsker (EP) estimate is

$$\hat{f}(x) = \sum_{j=j_0}^{\log_2(n/\ln(n))} \sum_m \sum_{k\in B_{j,m}} \hat{\theta}_{j,k}\psi_{j,k}(x), \quad x \in [0,1]. \tag{6}$$

A Stein estimate, corresponding to the shrinkage procedure (4), is defined as

$$\bar{f}_S(x) = \sum_{j=j_0}^{\log_2(n/\ln(n))} \sum_m \sum_{k\in B_{j,m}} \bar{\theta}_{j,k}\psi_{j,k}(x), \quad x \in [0,1]. \tag{7}$$

In what follows, an estimate (7) with the special James–Stein thresholds $\mu_{j,m} = -2/L_{j,m}$, $L_{j,m} > 2$ will be referred to as a James–Stein estimate $\bar{f}_{JS}$. The main difference between EP and Stein shrinkage procedures is that the former uses a hard thresholding while the latter a soft one. A minor difference is that Stein shrinkage may use non-positive thresholds; let us note that Stein estimates with positive thresholds may be also referred to as penalized Stein estimates.

3 Lower bounds for estimation of a spike function

We are using the Mean Squared Error (MSE) criteria for the analysis of blockwise–shrinkage estimates; the interested reader can find a discussion of this classical criteria in [8,11,15]. Because the class of estimates is specified, similarly to the methodology of a

numerical study we can choose a particular underlying regression function and then even assume that this function is known to the statistician. Keeping this in mind, the MSE is evaluated at a point $x_0 \in (0,1)$ and the estimated regression function will be a known (to the statistician) *spike* function

$$f_1(x) := \theta\psi_{i,r}(x). \tag{8}$$

Here $\theta > 0$ is a wavelet coefficient and i is a scale, $j_0 \le i \le \log_2(n/\ln(n))$ that will be specified later. The shift index $r := r(i, x_0, \psi)$ is defined from the relation $\psi^2_{i,r}(x_0) = \max_k \psi^2_{i,k}(x_0)$. Also, we define $B_{i,q}$ as a block to which r belongs, that is, $r \in B_{i,q}$ with $q := q(i, x_0, \psi)$.

Let us comment about the spike function (8). On finer scales this function does resemble a spike; see examples in [14]. This function is of a particular interest because typical signals considered in the statistical literature have sparsely located significant wavelet coefficients; see a discussion in [6,14]. Since on each scale several wavelet functions may be nonzero at the point x_0, it is natural to chose a wavelet function which takes on largest absolute value at this point.

Assumption 1. *A wavelet basis satisfies the assumption formulated in the first paragraph of Section 2, and also for some positive constant c^* the wavelet function ψ satisfies*

$$\max_k \min_x \psi^2(2^j x - k) \ge c^*/s, \;\; j \ge j_0, \tag{9}$$

where s is the rounded up support of the wavelet function ψ. Empirical wavelet errors $\{\xi_{j,k}\}$ are symmetrically distributed about zero, unit variance, independent and not necessarily identically distributed.

It is shown in [10] that assumption (9) holds for all wavelet functions traditionally used in statistical applications. Also note that under Assumption 1 the errors are not necessarily Gaussian (which is a traditional assumption in the regression literature) and this will make our results robust toward the distribution of empirical wavelet errors.

In what follows we shall need the following result established in [10].

Proposition 1. *(i) Suppose that Assumption 1 holds, consider estimation of the spike function (8) at the point x_0, and denote $\Theta := \theta^2/L_{i,q}$. Then the mean squared error of the oracle $\hat{f}^*$, defined in (5), can be calculated by the formula:*

$$E\{(\hat{f}^*(x_0) - f_1(x_0))^2\}$$
$$= \Big[\frac{\Theta^2 vn^{-1}}{(\Theta + vn^{-1})^2} + \frac{(vn^{-1})^2\theta^2}{(\Theta + vn^{-1})^2}\Big]\psi_{i,r}^2(x_0) + \frac{\Theta^2 vn^{-1}}{(\Theta + vn^{-1})^2}\sum_{k\in B_{i,q};k\neq r}\psi_{i,k}^2(x_0). \qquad (10)$$

(ii) If additionally $\theta^2 \le \lambda_{i,q}^2 L_{i,q} vn^{-1}$, then the mean squared errors of EP and Stein estimates, with $\mu_{j,m} = \lambda_{j,m}^2$, satisfy the lower bound

$$\min\Big(E\{(\hat{f}(x_0) - f_1(x_0))^2\}, E\{(\bar{f}_S(x_0) - f_1(x_0))^2\}\Big) \ge f_1^2(x_0)/4. \qquad (11)$$

Now let us present a new lower bound which improves known ones of [10].

Theorem 1. *Let Assumption 1 hold and additionally $\{\xi_{j,k}\}$ have uniformly bounded fourth moments. Suppose that $\theta^2 > 2\lambda_{i,q}^2 L_{i,q} vn^{-1}$ for EP estimate, $\theta^2 > 2\mu_{i,q} L_{i,q} vn^{-1}$ for Stein estimate and $\theta^2 > 0$ for James–Stein estimate. Then there exists a constant $c_1 < \infty$, depending only on $\max_{j,k} E\xi_{j,k}^4$ and the support of the wavelet function ψ, such that the following lower–bound oracle inequality holds for EP, Stein and James–Stein estimates:*

$$\min\Big(E\{(\hat{f}(x_0) - f_1(x_0))^2\}, E\{(\bar{f}_S(x_0) - f_1(x_0))^2\}, E\{(\bar{f}_{JS}(x_0) - f_1(x_0))^2\}\Big)$$
$$\ge E\{(\hat{f}^*(x_0) - f_1(x_0))^2\}\Big[1 - c_1 L_{i,q}^{-1/2}\Big(1 + \nu_{SJS}|\mu_{i,q}|\Big)$$
$$\times\Big(\frac{(\Theta + vn^{-1})^{1/2}}{(vn^{-1})^{1/2}} + \frac{\Theta + vn^{-1}}{L_{i,q}^{1/2} vn^{-1}} + \frac{(\Theta + vn^{-1})^{1/2}}{(L_{i,q}\Theta)^{1/2}}\Big)\Big]. \qquad (12)$$

Here $\nu_{SJS} = 0$ for EP estimate and $\nu_{SJS} = 1$ for Stein or James–Stein estimate.

Note that the lower bound (12) can be called a lower–bound oracle inequality because the oracle's MSE is the factor in this lower bound. Let us recall that an oracle inequality traditionally means an upper bound which contains, as a factor, an oracle's risk. As we shall see, lower–bound oracle inequalities nicely complement the traditional upper–bound ones.

Now we are in a position to present main results of this article, namely several universal corollaries that, in particular, will allow us to consider a non–asymptotic minimax lower bound. Define a function class $\mathcal{H}_w(\alpha,\delta,x_0)$, $\alpha>0$, $\delta>0$ such that wavelet coefficients of a function from this class satisfy the inequality

$$\max_j \max_{k\in K(x_0,\delta,j)} |\theta_{j,k}| 2^{j(\alpha+1/2)} \le C_{\mathcal{H}}. \tag{13}$$

Here $K(x_0,\delta,j)$ is a set of indices k such that intersection of $(x_0-\delta,x_0+\delta)$ with the support of $\psi(2^j x_0-k)$ is not empty, and $C_{\mathcal{H}}:=C(\alpha,\delta,x_0)<\infty$. This class is introduced and discussed in [10], and in what follows we need only to know that if the wavelet function ψ is sufficiently smooth then this class is a subset of a classical local Hölder class of order α (thus all minimax lower bounds established for $\mathcal{H}_w(\alpha,\delta,x_0)$ also hold for the Hölder class), and that the corresponding adaptive minimax rate of the MSE convergence for this class is

$$R_{\mathcal{H}}:=(\ln(n)n^{-1})^{2\alpha/(2\alpha+1)}. \tag{14}$$

Note that only the parameter α affects the adaptive minimax rate; a discussion of (14) can be found in [8,14].

Now we can introduce a "universal" spike function. Define a scale

$$\tau:=\max\Big\{j:\quad 2^j\le\Big[\frac{C_{\mathcal{H}}^2}{2b_n\ln(n)vn^{-1}}\Big]^{1/(2\alpha+1)},\ j_0\le j\le\log_2(n/\ln(n))\Big\}, \tag{15}$$

where a positive sequence b_n is such that $b_n\to\infty$ as slowly as desired when $n\to\infty$, and introduce

$$f_{1n}(x):=(b_n\ln(n)vn^{-1})^{1/2}\psi_{\tau,r}(x). \tag{16}$$

It is easy to verify that both f_{1n} and $\sqrt{2}f_{1n}$ belong to $\mathcal{H}_w(\alpha,\delta,x_0)$, that is, they satisfy (13). Also denote $C_1:=(C_{\mathcal{H}}^2v^{2\alpha}/2)^{1/(2\alpha+1)}(c^*/8s)$. Let us stress that the spike function (16) does not depend on a considered wavelet estimate; the interested reader can compare it with spike functions studied in [10] which crucially depend on blocks and thresholds of analyzed wavelet estimates.

Corollary 1. *Suppose that Assumption 1 holds, and consider the problem of estimation of the spike function $f_{1n}(x)$ under the MSE criteria at the point x_0. Introduce a class $\mathcal{E}_1$ of EP and Stein estimates satisfying $\lambda^2_{\tau,q}L_{\tau,q} \geq b_n \ln(n)$ and $\mu_{\tau,q}L_{\tau,q} \geq b_n \ln(n)$, respectively. Then,*

$$\inf_{\check{f}\in\mathcal{E}_1} E\{(\check{f}(x_0) - f_{1n}(x_0))^2\} \geq C_1(b_n \ln(n)n^{-1})^{2\alpha/(2\alpha+1)}. \tag{17}$$

Corollary 1 allows us to analyze blockwise–shrinkage estimates with relatively large thresholds and/or blocks. There are suggested vast portfolios of blockwise–shrinkage estimates that are covered by this corollary; see [2,3,6,8]. On the other hand, many known estimates, including James–Stein one, cannot be analyzed via (17). Our next corollary is devoted to a complementary class of blockwise–shrinkage estimates that may include relatively small and even negative thresholds.

Corollary 2. *Suppose that Assumption 1 holds, fourth moments of the errors $\{\xi_{j,k}\}$ are uniformly bounded, and consider the problem of estimation of the spike function $\sqrt{2}f_{1n}(x)$ at the point x_0. Introduce a class $\mathcal{E}_2$ of EP, Stein and James–Stein estimates such that $\lambda^2_{\tau,q}L_{\tau,q} < b_n \ln(n)$ and $\mu_{\tau,q}L_{\tau,q} < b_n \ln(n)$ for EP and Stein/James–Stein estimates, respectively, and let additionally $L_{\tau,q} \geq b_n \ln(n)$ for all these estimates. Then*

$$\inf_{\check{f}\in\mathcal{E}_2} E\{(\check{f}(x_0) - f_{1n}(x_0))^2\} \geq C_1(1-\gamma_n)(b_n \ln(n)n^{-1})^{2\alpha/(2\alpha+1)}, \tag{18}$$

where $\gamma_n := (b_n \ln(n))^{-1}[c_1(18 + 2(b_n \ln(n))^{-1/2}) + 9]$.

We shall discuss the obtained results in the next section.

4 Discussion

1. This article is solely devoted to the study of blockwise–shrinkage estimates. Let us explain why these estimates have such a prominent place in the statistical literature and why the study of these estimates is a hot topic in the literature. First of all, they have excellent asymptotic properties of being adaptive minimax under global and local

criterion like MISE and MSE and over vast function classes of both smooth and spatially inhomogeneous functions. Second, a typical adaptive nonparametric estimator, like kernel or spline one, requires a numerical optimization; blockwise–shrinkage estimates employ no optimization at all. Third, while wavelet blockwise–shrinkage estimates are probably more popular among applied statisticians, the same algorithm can be applied to any orthogonal basis including trigonometric, polynomial and multiwavelet; see a discussion in [8]. Fourth, these adaptive estimates are known for their property to mimic performance of oracles and being superefficient; see [1,2,8]. Finally, in the theory of estimation of functionals of a regression function and solving inverse problems, the estimate is known as an optimal plug–in one. We can conclude that blockwise–shrinkage estimates possess a unique bouquet of statistical properties unknown for other statistical procedures.

2. Let us show that the obtained results, due to the universal spike function, easily imply nonasymptotic minimax results. As an example, let us consider the lower bound (18). It was noticed earlier that the spike function $\sqrt{2}f_{1n}$ is Hölder of order α, it belongs to $\mathcal{H}_w(\alpha, \delta, x_0)$, and this spike obviously depends on neither an estimated regression function nor an employed blockwise–shrinkage estimate. This immediately yields the following minimax lower bound:

$$\inf_{\check{f}\in\mathcal{E}_2} \sup_{f\in\mathcal{H}_w(\alpha,\delta,x_0)} E\{(\check{f}(x_0) - f(x_0))^2\} \ge \inf_{\check{f}\in\mathcal{E}_2} E\{(\check{f}(x_0) - \sqrt{2}f_{1n}(x_0))^2\}$$

$$\ge C_1(1-\gamma_n)(b_n \ln(n) n^{-1})^{2\alpha/(2\alpha+1)}. \tag{19}$$

Thus, any blockwise–shrinkage estimate from the class $\mathcal{E}_2$ is not adaptive rate optimal in the sense of not attaining the adaptive minimax rate (14).

As an example, let us compare this result with the well–known (and established in a traditional manner) minimax result of [2]. In that article Stein estimates with constant thresholds and blocks of identical length were studied, and it was established that if length of blocks is proportional to $[\ln(n)]^{1+\nu}$, $\nu > 0$ then the minimax MSE of this estimate over a Hölder class of order α could not attain (14). This was a pioneering result in the study of blockwise–shrinkage estimates via the MSE criteria. The lower bound (19) extends

that result in three important directions. First of all, a dramatically larger class of blocks and thresholds is considered, in particular thresholds are not necessarily constant and let us recall that the sequence b_n can increase as slowly as desired (not necessarily as a logarithmic function). Second, a minimax approach does not explain what are the functions that are most difficult for the estimation. On the other hand, the lower bound (19) points upon the spike function which prevents all estimates from the considered class $\mathcal{E}_2$ to attain the optimal rate (14). Finally, the lower bound is nonasymptotic.

In a similar way the interested reader can deduce a minimax lower bound from (17). Note that typically a lower minimax bound is the most technically involved part of a nonparametric regression analysis. The suggested methodology of obtaining a lower minimax bound is probably the simplest one known in the literature.

3. The practical and pedagogic value of the minimax result (19) is as follows. In a minimax game between the nature and the statistician, the nature chooses a function and then the statistician observes corresponding empirical wavelet coefficients. A typical nature's strategy, implying a worst case scenario for the statistician, is to choose the function at random from a set of functions known to the statistician; see a discussion in [8,11]. The interesting feature of the game (19) is that, whenever the statistician uses a blockwise–shrinkage estimate from the class $\mathcal{E}_2$, the nature's choice is a fixed spike function *known* to the statistician. In other words, the nature plays with open cards. Such a game is known, for instance, in the parametric theory of uniformly minimum variance unbiased estimates, but it is new in the nonparametric literature. On the other hand, the nature's strategy resembles the methodology employed in traditional numerical studies of wavelet estimates where performance of estimates is studied for a finite set of estimated functions.

4. Blockwise–shrinkage estimates are traditionally studied under the MISE criteria. Under this criteria, there is a huge pool of blocks–thresholds–shrinkages that imply sharp minimax convergence of the MISE. The interested reader can find a discussion in [3,7,8]. As we have seen, using the MSE criteria allows the statistician to dramatically shrink this pool of estimates but it still remains relatively large. It is an open problem to add

an extra criteria (or criterion) that will lead to a single optimal estimate. A discussion of this problem can be found in [2] where an optimal Stein estimator for a class of constant thresholds and blocks with identical lengths depending only on the sample size is suggested. An interesting numerical study and a discussion of Bayesian ideas can be found in [5].

5. The interested reader can use software package of [8] to test performance of different wavelet estimates on a spike function. Several numerical examples and results of intensive Monte Carlo experiments can be found in [10].

6. It is an interesting and open problem to extend the obtained results upon more general regression settings (like the one with not uniform design considered in [4]) as well as upon other nonparametric problems (like the probability or spectral density estimation) traditionally considered in the wavelet literature.

5 Proofs

Proof of Theorem 1. Denote $\Lambda := \Theta/(\Theta + vn^{-1})$, $\hat{\Theta}_{j,m} = \sum_{k\in B_{j,m}}(\tilde{\theta}^2_{j,k} - vn^{-1})$, and

$$\hat{\Lambda}_{j,m} := \frac{\sum_{k\in B_{j,m}}(\tilde{\theta}^2_{j,k} - vn^{-1})}{\sum_{k\in B_{j,m}}\tilde{\theta}^2_{j,k}} I(\sum_{k\in B_{j,m}} \tilde{\theta}^2_{j,k} > (1+\lambda^2_{j,m})L_{j,m}vn^{-1}).$$

Write

$$E(\hat{f}(x_0) - f_1(x_0))^2 = E(\hat{f}(x_0) - \hat{f}^*(x_0) + \hat{f}^*(x_0) - f_1(x_0))^2$$

$$\geq E(\hat{f}^*(x_0) - f_1(x_0))^2 + 2E(\hat{f}(x_0) - \hat{f}^*(x_0))(\hat{f}^*(x_0) - f_1(x_0)).$$

The first term can be evaluated using (10). Write,

$$E(\hat{f}(x_0) - \hat{f}^*(x_0))(\hat{f}^*(x_0) - f_1(x_0))$$

$$= vn^{-1}\Lambda_{i,q}E\{(\hat{\Lambda}_{i,q} - \Lambda_{i,q})[\sum_{k\in T_{i,q}} \xi_{i,k}\psi_{i,k}(x_0)]^2\} + (vn^{-1})^{1/2}\theta\Lambda_{i,q}\psi^2_{i,r}(x_0)E\{(\hat{\Lambda}_{i,q} - \Lambda_{i,q})\xi_{i,r}\}$$

$$+\theta^2(\Lambda_{i,q} - 1)\psi^2_{i,r}(x_0)E(\hat{\Lambda}_{i,q} - \Lambda_{i,q}) + (vn^{-1})^{1/2}(\Lambda_{i,q} - 1)\theta\psi_{i,r}(x_0)E\{(\hat{\Lambda}_{i,q} - \Lambda_{i,q})\xi_{i,r}\psi_{i,r}(x_0)\}$$

$$=: T_1 + T_2 + T_3 + T_4. \tag{20}$$

Now we recall two familiar relationships (see, for instance, (34)–(35) in [10])

$$|\hat{\Lambda}_{i,q} - \Lambda_{i,q}| \le 2\frac{|\hat{\Theta}_{i,q} - \Theta_{i,q}|}{\Theta_{i,q} + vn^{-1}}. \tag{21}$$

and

$$E(\hat{\Theta}_{j,m} - \Theta_{j,m})^2 \le c_2 L_{j,m}^{-1} vn^{-1}(4\Theta_{j,m} + 2vn^{-1}). \tag{22}$$

Here c_2 is a finite constant that depends only on $\max_{i,k} E\xi_{i,k}^4$; moreover $c_2 = 1$ for Gaussian empirical errors $\{\xi_{j,k}\}$ and in this case we have equality in (22).

Using these results to estimate T_1 we get (in what follows C's denote generic positive constants)

$$T_1 = vn^{-1}\Lambda_{i,q} E\{(\hat{\Lambda}_{i,q} - \Lambda_{i,q}) \sum_{k \in B_{i,q}} \xi_{i,k}^2 \psi_{i,k}^2(x_0)\}$$

$$\ge -vn^{-1}\Lambda_{i,q} \sum_{k \in B_{i,q}} \psi_{i,k}^2(x_0) E^{1/2}(\hat{\Lambda}_{i,q} - \Lambda_{i,q})^2 E^{1/2}\{[\max_{k:\, \psi_{i,k}(x_0) \ne 0} \xi_{i,k}^2]^2\}$$

$$> -Cvn^{-1}\Lambda_{i,q} \sum_{k \in B_{i,q}} \psi_{i,k}^2(x_0) \frac{L_{i,q}^{-1/2}(vn^{-1})^{1/2}(\Theta + vn^{-1})^{1/2}}{\Theta + vn^{-1}}$$

$$> \frac{-(vn^{-1})^2\theta^2\psi_{i,r}^2(x_0)}{(\Theta + vn^{-1})^2}\Big[\frac{C(\Theta + vn^{-1})^{1/2}}{L_{i,q}^{3/2}(vn^{-1})^{1/2}}\Big].$$

In the last inequality we used $\sum_{k \in B_{i,q}} \psi_{i,k}^2(x_0) \le s\psi_{i,r}^2(x_0)$ based on the definition of r and the supposed bounded support of the wavelet function ψ.

To evaluate T_2, T_3 and T_4 we may use the following known lower bounds of [10]:

$$T_2 \ge -C\frac{(vn^{-1})^2\theta^2\psi_{i,r}^2(x_0)}{(\Theta + vn^{-1})^2}[\frac{\Theta + vn^{-1}}{L_{i,q}vn^{-1}}],$$

$$T_3 \ge -C\frac{(vn^{-1})^2\theta^2\psi_{i,r}^2(x_0)}{(\Theta + vn^{-1})^2}[L_{i,q}^{-1/2}\frac{(\Theta + vn^{-1})^{1/2}}{(vn^{-1})^{1/2}}],$$

and

$$T_4 \ge -C\frac{(vn^{-1})^2\theta^2\psi_{i,r}^2(x_0)}{(\Theta + vn^{-1})^2}[\frac{(\Theta + vn^{-1})^{1/2}}{L_{i,q}\Theta^{1/2}}].$$

Combining the results and using (10) we get

$$E(\hat{f}(x_0) - f_1(x_0))^2 \ge E(\hat{f}^*(x_0) - f_1(x_0))^2$$
$$\times\Big[1 - CL_{i,q}^{-1/2}\Big(\frac{(\Theta + vn^{-1})^{1/2}}{(vn^{-1})^{1/2}} + \frac{\Theta + vn^{-1}}{L_{i,q}^{1/2}vn^{-1}} + \frac{(\Theta + vn^{-1})^{1/2}}{(L_{i,q}\Theta)^{1/2}}\Big)\Big].$$

This verifies (12) for the case of EP estimate. Now we are considering the case of Stein and James-Stein estimates. The difference between these two estimates and EP estimate is in the using of a specific shrinkage factor

$$\bar{\Lambda}_{i,q} := \frac{\hat{\Theta}_{i,q} - \mu_{i,q}vn^{-1}}{\hat{\Theta}_{i,q} + vn^{-1}} I(\hat{\Theta}_{i,q} > \mu_{i,q}vn^{-1})$$

in place of $\hat{\Lambda}_{i,q}$. As a result, to make the proof shorter by employing the previous proof, we need to modify (21)-(22) for the considered case. Write

$$\bar{\Lambda}_{i,q} - \Lambda_{i,q} = \Big[\frac{vn^{-1}(\hat{\Theta}_{i,q} - \Theta_{i,q})}{(\hat{\Theta}_{i,q} + vn^{-1})(\Theta_{i,q} + vn^{-1})} I(\hat{\Theta}_{i,q} > \mu_{i,q}vn^{-1})$$
$$- \frac{\Theta_{i,q}}{\Theta_{i,q} + vn^{-1}} I(\hat{\Theta}_{i,q} \le \mu_{i,q}vn^{-1})\Big] + \Big[-\frac{\mu_{i,q}vn^{-1}}{\hat{\Theta}_{i,q} + vn^{-1}} I(\hat{\Theta}_{i,q} > \mu_{i,q}vn^{-1})\Big]$$
$$=: A_1 + A_2. \tag{23}$$

The term A_1 has been studied in the earlier proof for EP estimate; it gives us the special addend in (12) that corresponds to $\nu_{SJS} = 0$. The term A_2 explains the other extra addends in (12). Let us denote by T_1', T_2', T_3' and T_4' the corresponding addends for $T_1, \dots, T_4$ caused by A_2 and then evaluate them in turn. Write

$$T_1' := vn^{-1}\Lambda_{i,q}E\{A_2[\sum_{k \in B_{i,q}} \xi_{i,k}\psi_{i,k}(x_0)]^2\}.$$

Note that $T_1' \ge 0$ for James–Stein estimate, hence we need to find a lower bound for T_1' only for the case of Stein estimate, that is, for the case $\mu_{i,q} \ge 0$. Then a straightforward calculation yields

$$T_1' \ge -\frac{(vn^{-1})^2\theta^2\psi_{i,r}^2(x_0)}{(\Theta + vn^{-1})^2}\Big[\frac{C\mu_{i,q}}{L_{i,q}}\Big(1 + \frac{(\Theta + vn^{-1})^{1/2}}{L_{i,q}^{1/2}(vn^{-1})^{1/2}}\Big)\Big].$$

Similarly we obtain

$$T_2' \geq -\frac{(vn^{-1})^2\theta^2\psi_{i,r}^2(x_0)}{(\Theta+vn^{-1})^2}\Big[\frac{C|\mu_{i,q}|(\Theta+vn^{-1})}{L_{i,q}^{1/2}vn^{-1}}\Big].$$

The third term is

$$T_3' := \theta^2(\Lambda_{i,q}-1)\psi_{i,r}^2(x_0)E\{-\frac{\mu_{i,q}vn^{-1}}{\hat{\Theta}+vn^{-1}}I(\hat{\Theta}_{i,q}>\mu_{i,q}vn^{-1})\}.$$

It is nonnegative for Stein estimate because $\Lambda_{i,q}-1<0$. For James–Stein estimate we get

$$T_3' \geq -\frac{(vn^{-1})^2\theta^2\psi_{i,r}^2(x_0)}{(\Theta+vn^{-1})^2}\Big[\frac{|\mu_{i,q}|(\Theta+vn^{-1})}{(1+\mu_{i,q})vn^{-1}}\Big] \geq -\frac{(vn^{-1})^2\theta^2\psi_{i,r}^2(x_0)}{(\Theta+vn^{-1})^2}\Big[\frac{6(\Theta+vn^{-1})}{L_{i,q}vn^{-1}}\Big].$$

In the last inequality we used the assumption $L_{i,q}>2$ for James-Stein estimate.

The last term to consider is

$$T_4' := (vn^{-1})^{1/2}(\Lambda_{i,q}-1)\theta\psi_{i,r}(x_0)E\Big\{-\frac{\mu_{i,q}vn^{-1}}{\hat{\Theta}_{i,q}+vn^{-1}}I(\hat{\Theta}_{i,q}>\mu_{i,q}vn^{-1})\xi_{i,r}\psi_{i,r}(x_0)\Big\}.$$

A simple analysis of this term reveals that it is positive for Stein estimate. Thus, only the case of James–Stein estimate should be considered. Write,

$$T_4' = \frac{(vn^{-1})^{-5/2}\theta\psi_{i,r}^2(x_0)\mu_{i,q}}{\Theta+vn^{-1}}E\{\frac{\xi_{i,r}}{\hat{\Theta}+vn^{-1}}I(\hat{\Theta}_{i,q}>\mu_{i,q}vn^{-1})\}$$

$$= \frac{(vn^{-1})^2\theta^2\psi_{i,r}^2(x_0)}{(\Theta+vn^{-1})^2}\Big[\frac{(vn^{-1})^{1/2}(\Theta+vn^{-1})\mu_{i,q}}{\theta}E\{\frac{\xi_{i,r}}{\hat{\Theta}_{i,q}+vn^{-1}}I(\hat{\Theta}_{i,q}>\mu_{i,q}vn^{-1})\}\Big].$$

Consider the last expectation,

$$E\{\frac{\xi_{i,r}}{\hat{\Theta}_{i,q}+vn^{-1}}I(\hat{\Theta}_{i,q}>\mu_{i,q}vn^{-1})\} = \frac{1}{\Theta+vn^{-1}}E\{\xi_{i,r}I(\hat{\Theta}_{i,q}>\mu_{i,q}vn^{-1})\}$$

$$+E\{\frac{(\Theta-\hat{\Theta}_{i,q})\xi_{i,r}}{(\hat{\Theta}_{i,q}+vn^{-1})(\Theta+vn^{-1})}I(\hat{\Theta}_{i,q}>\mu_{i,q}vn^{-1})\}.$$

Using this together with $\theta^2 = L_{i,q}\Theta$ we get:

$$T_4' \geq -\frac{(vn^{-1})^2\theta^2\psi_{i,r}^2(x_0)}{(\Theta+vn^{-1})^2}\Big[\frac{(vn^{-1})^{1/2}|\mu_{i,q}|}{L_{i,q}^{1/2}\Theta^{1/2}}$$

$$+\frac{(vn^{-1})^{1/2}(\Theta+vn^{-1})|\mu_{i,q}|c_2^{1/2}L_{i,q}^{-1/2}(vn^{-1})^{1/2}(4\Theta+2vn^{-1})^{1/2}}{L_{i,q}^{1/2}\Theta^{1/2}(1+\mu_{i,q})(vn^{-1})(\Theta+vn^{-1})}\Big]$$

$$\ge -\frac{(vn^{-1})^2\theta^2\psi_{i,r}^2(x_0)}{(\Theta+vn^{-1})^2}\Big[\frac{C|\mu_{i,q}|(\Theta+vn^{-1})^{1/2}}{L_{i,q}^{1/2}\Theta^{1/2}}\Big].$$

Recall that $\mu_{i,q}=-2/L_{i,q}$ for James–Stein estimate and thus

$$T_4' \ge -\frac{(vn^{-1})^2\theta^2\psi_{i,r}^2(x_0)}{(\Theta+vn^{-1})^2}\Big[\frac{C(\Theta+vn^{-1})^{1/2}}{L_{i,q}^{3/2}\Theta^{1/2}}\Big].$$

Combining the results we get (in what follows ν_{JS} is the indicator of the James–Stein estimate)

$$T_1'+T_2'+T_3'+T_4' \ge -\frac{(vn^{-1})^2\theta^2\psi_{i,r}^2(x_0)}{(\Theta+vn^{-1})^2}$$

$$\times\Big[C|\mu_{i,q}|\Big(L_{i,q}^{-1}(1+\frac{(\Theta+vn^{-1})^{1/2}}{L_{i,q}^{1/2}(vn^{-1})^{1/2}})+\frac{\Theta+vn^{-1}}{L_{i,q}^{1/2}vn^{-1}}\Big)+\nu_{JS}\Big(C\frac{(\Theta+vn^{-1})^{1/2}}{L_{i,q}^{3/2}\Theta^{1/2}}+\frac{6(\Theta+vn^{-1})}{L_{i,q}vn^{-1}}\Big)\Big]$$

$$\ge -\frac{(vn^{-1})^2\theta^2\psi_{i,r}^2(x_0)}{(\Theta+vn^{-1})^2}\Big[\frac{C|\mu_{i,q}|(\Theta+vn^{-1})}{L_{i,q}^{1/2}vn^{-1}}+\nu_{JS}\Big(C\frac{(\Theta+vn^{-1})^{1/2}}{L_{i,q}^{3/2}\Theta^{1/2}}+\frac{6(\Theta+vn^{-1})}{L_{i,q}vn^{-1}}\Big)\Big].$$

This together with (10) finishes the proof of (12). Theorem 1 is verified.

Proof of Corollary 1. It is plain to check that the spike function $f_{1n}(x) =: \theta\psi_{\tau,r}(x)$, $\theta^2 = b_n\ln(n)vn^{-1}$ satisfies the assumption of Proposition 1. Indeed, $\theta^2 \le \lambda_{\tau,q}^2 L_{\tau,q}vn^{-1}$ and $\theta^2 \le \mu_{\tau,q}^2 L_{\tau,q}vn^{-1}$ for EP and Stein estimates from the class $\mathcal{E}_1$, respectively. Then a direct calculation shows that

$$f_{1n}^2(x_0)/4 = (1/4)b_n\ln(n)vn^{-1}2^\tau\psi^2(2^\tau x_0 - r)$$

$$\ge (1/4)b_n\ln(n)vn^{-1}(1/2)\Big[\frac{C_{\mathcal{H}}^2}{2b_n\ln(n)vn^{-1}}\Big]^{1/(2\alpha+1)}(c^*/s)$$

$$= C_{\mathcal{H}}^{2/(2\alpha+1)}(c^*/8s)(1/2)(2b_n\ln(n)vn^{-1})^{2\alpha/(2\alpha+1)} = C_1(b_n\ln(n)n^{-1})^{2\alpha/(2\alpha+1)}.$$

This proves Corollary 1.

Proof of Corollary 2. It is easy to check that the considered spike function $\sqrt{2}f_{1n}(x) = \theta\psi_{\tau,r}(x)$, $\theta^2 := 2b_n \ln(n)vn^{-1}$ satisfies the assumption of Theorem 1. Then (10) and a direct calculation show that for the considered class $\mathcal{E}_2$ of estimates we can write

$$E(\hat{f}^*(x_0) - \sqrt{2}f_{1n}(x_0))^2 \ge \frac{(vn^{-1})^2\theta^2}{(\theta^2/L_{\tau,q} + vn^{-1})^2}2^\tau\psi^2(2^\tau x_0 - r)$$

$$\ge \frac{2b_n \ln(n)vn^{-1}}{(1 + 2b_n \ln(n)/L_{\tau,q})^2}(1/2)\Big[\frac{C_{\mathcal{H}}^2}{2b_n \ln(n)vn^{-1}}\Big]^{1/(2\alpha+1)}(c^*/s)$$

$$\ge C_{\mathcal{H}}^{2/(2\alpha+1)}(1/18)(2b_n \ln(n)vn^{-1})^{2\alpha/(2\alpha+1)}(c^*/s)$$

$$\ge C_1(8/18)2(b_n \ln(n)n^{-1})^{2\alpha/(2\alpha+1)}.$$

Also a direct calculation shows that

$$\frac{\Theta + vn^{-1}}{vn^{-1}} = \frac{2b_n \ln(n)}{L_{\tau,q}} + 1 \le 3,$$

and

$$\frac{\Theta + vn^{-1}}{L_{\tau,q}\Theta} = \frac{2b_n \ln(n)L_{\tau,q}^{-1} + 1}{2b_n \ln(n)} = \frac{1}{L_{\tau,q}} + \frac{1}{2b_n \ln(n)} \le \frac{3}{2b_n \ln(n)}.$$

Another plain observation is that $\mu_{\tau,q} \le b_n \ln(n)/L_{\tau,q} < 1$ for the considered Stein estimates. Using these facts, we note that the factor F in square brackets of (12) can be evaluated from below as follows:

$$F \ge 1 - c_1\Big(1 + \frac{b_n \ln(n)}{L_{\tau,q}}\Big)\Big(\frac{3^{1/2}}{L_{\tau,q}} + \frac{3}{2b_n \ln(n)} + \frac{3^{1/2}}{(L_{\tau,q}2b_n \ln(n))^{1/2}}\Big)$$

$$-\frac{c_1 3^{1/2}}{L_{\tau,q}(2b_n \ln(n))^{1/2}} - \frac{9}{b_n \ln(n)} \ge 1 - \frac{c_1(18 + 2(b_n \ln(n))^{-1/2}) + 9}{b_n \ln(n)}.$$

Combining the results we prove Corollary 2.

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Construction of Biorthogonal B-spline Type Wavelet Sequences with Certain Regularities

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Abstract

In this paper we study a refinement equation of the form $\phi(x) = \sum_{n\in Z} h_n\phi(2x-n)$, where $\{h_n\}$ is a finitely supported sequence. Let the symbol of the sequnce $m(z) := \frac{1}{2}\sum_n h_n z^n$ take the form $m(z) = (\frac{1+z}{2})^N z^{k'}\sum_{j=0}^k a_j z^j$ where $\sum_j a_j = 1$ and $\sum_j (-1)^j a_j \neq 0$. Under the assumption $\left(\sum_{0\leq odd\, j\leq k}|a_j|\right)^2 + \left(\sum_{0\leq even\, j\leq k}|a_j|\right)^2 < 2^{2N-1}$ we show that the corresponding ϕ is in L_2. Then the B-spline type wavelet sequences that possess the largest possible regularities and required vanishing moments are characterized.

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1 Introduction

We start by setting some notations. We define a *low-pass filter* as

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$$m_0(\xi) = 2^{-1} \sum_n c_n e^{-in\xi}. \tag{1.1}$$

Here, we assume that only finitely many c_n are nonzero. However, some of our results can be extended to infinite sequences that have sufficient decay for $|n| \to \infty$. Next, we define ϕ by

$$\hat{\phi}(\xi) = \Pi_{j=1}^{\infty} m_0(2^{-j}\xi). \tag{1.2}$$

This infinite product converges only if $m_0(0) = 1$; i.e., if $\sum_n c_n = 2$. In this case, the infinite products in (2) converge uniformly and absolutely on compact sets, so that $\hat{\phi}$ is a well-defined C^∞ function. Obviously, $\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2)$, or, equivalently, $\phi(t) = \sum_n c_n \phi(2t - n)$ at least in the sense of distributions. From Lemma 3.1 in [2], ϕ has compact support.

We now consider the simplest possible masks $m_0(\xi)$ with the following form.

Definition 1.1 *Denote by Φ the set of all B-spline type scaling functions $\phi(t)$ that have Fourier transform $\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2)$. Here the filter*

$$m_0(\xi) = 2^{-1} \sum_n c_n e^{-in\xi}$$

is in the set M that contains all filters with the form

$$m_0^N(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)^N F(\xi), \tag{1.3}$$

where

$$F(\xi) = e^{-ik'\xi} \sum_{j=0}^{k} a_j e^{-ij\xi}. \tag{1.4}$$

Here, all coefficients of $F(\xi)$ are real, $F(0) = 1$; N and k are positive integers; and $k' \in \mathbb{Z}$. Hence, the corresponding ϕ can be written as

$$\hat{\phi}(\xi) = \left(\frac{1 + e^{-i\xi/2}}{2}\right)^N F(\xi/2)\hat{\phi}(\xi/2). \tag{1.5}$$

Clearly, ϕ is a B-spline of order N if $F(\xi) = 1$. Thus, we call ϕ as defined by Def. 1.1 a B-spline type scaling function. The vanishing moments of ϕ are completely controlled by the exponents of its "spline factor," $\left(\frac{1+e^{-i\xi}}{2}\right)^N$. In addition, the regularity of ϕ is justified by the factors $F(\xi)$, and is independent of its vanishing moment.

Let both ϕ and $\tilde{\phi}$ be B-spline type scaling functions defined by Def. 1.1. Then

$$\hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2), \quad \hat{\tilde{\phi}}(\xi) = \tilde{m}_0(\xi/2)\hat{\tilde{\phi}}(\xi/2),$$

or, equivalently,

$$\phi(t) = \sum_n c_n \phi(2t-n), \quad \tilde{\phi}(t) = \sum_n \tilde{c}_n \phi(2t-n),$$

at least in the sense of distributions. Here, both $m_0(\xi) \in M$ and $\tilde{m}_0(\xi) \in M$ satisfy Eqs. (1.3) and (1.4). Thus,

$$m_0(\xi) = m_0^N(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N F(\xi) \tag{1.6}$$

$$\tilde{m}_0(\xi) = \tilde{m}_0^{\tilde{N}}(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^{\tilde{N}} \tilde{F}(\xi), \tag{1.7}$$

where

$$F(\xi) = e^{-ik'\xi} \sum_{j=0}^{k} a_j e^{-ij\xi} \quad \tilde{F}(\xi) = e^{-i\tilde{k}'\xi} \sum_{j=0}^{\tilde{k}} \tilde{a}_j e^{-ij\xi}.$$

From Lemma 3.1 in [2], ϕ and $\tilde{\phi}$ have compact support.

We also define the corresponding ψ and $\tilde{\psi}$ by

$$\hat{\psi}(\xi) = e^{i\xi/2}\overline{\tilde{m}_0(\xi/2+\pi)}\hat{\phi}(\xi/2), \quad \hat{\tilde{\psi}}(\xi) = e^{i\xi/2}\overline{m_0(\xi/2+\pi)}\hat{\phi}(\xi/2), \tag{1.8}$$

or, equivalently,

$$\psi(x) = \sum_n (-1)^{n-1} \tilde{c}_{-n-1} \phi(2x-n). \quad \tilde{\psi}(x) = \sum_n (-1)^{n-1} c_{-n-1} \phi(2x-n). \tag{1.9}$$

Since vanishing moment conditions $\int x^\ell \psi(x)dx = 0$, $\ell = 0, 1, \cdots, L$, are equivalent to $\frac{d^\ell}{d\xi^\ell}\hat{\psi}|_{\xi=0} = 0$, $\ell = 0, 1, \cdots, L$, we immediately know that the maximum number of vanishing moments for ψ and $\tilde{\psi}$ are $N-1$ and $\tilde{N}-1$, respectively. Therefore, the vanishing moments of ϕ and $\tilde{\phi}$ are completely determined by the exponent of their "spline factors" $\left(\frac{1+e^{-i\xi}}{2}\right)^N$ and $\left(\frac{1+e^{-i\xi}}{2}\right)^{\tilde{N}}$. In addition, as we pointed out before, the regularities of ϕ and $\tilde{\phi}$ will be justified by factors $F(\xi)$ and $\tilde{F}(\xi)$ and are independent of their vanishing moments.

Ingrid Daubechies, in her book "Ten Lectures on Wavelets," wrote: "What is more important, vanishing moments or regularity? The answer depends on the application, and is not always clear." She also pointed out that achieving higher regularity by increasing vanishing moments is not efficient, because 80% of the zero moments are wasted. In [He1998], we gave a method for constructing biorthogonal wavelets with the largest possible regularities and required vanishing moments based on the established condition. In the next section, we give a condition for the coefficients a_j $(j = 0, 1, \cdots, k)$ of $F(\xi)$ such that the corresponding ϕ is in $L_2(\mathbb{R})$. By using the condition we improve the results in [6]. In Section 3, we will give a method for constructing a sequence of B-spline type scaling and wavelets from either an orthogonal sacaling function or a pair of biorthogonal scalings by using their convolutions with certain B-splines. In particular, if both the method and the lifting scheme of Sweldens (see [11]) are applied, then all of the pairs of biorthogonal spline type scaling functions shown in references [2] and [4] can be constructed from the Haar scaling function.

2 Construction of biorthogonal B-spline type wavelets

Denote

$$\sum_{k;even\ j} b_j \equiv \sum_{0\le even\ j\le k} b_j \qquad\qquad \sum_{k,odd\ j} b_j \equiv \sum_{0<odd\ j\le k} b_j.$$

We now establish the following main results of this section.

Theorem 2.1 *Let $\phi \in \Phi$ be defined as in Definition 1.1; i.e.,*

$$\phi = \Pi_{j=1}^{\infty} m_0^N(2^{-j}\xi),$$

where $m_0^N(\xi) \in M$ is defined by (1.3) and (1.4):

$$m_0^N(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N F(\xi)$$

and $F(\xi) = e^{-ik'\xi}\sum_{j=0}^{k} a_j e^{-ij\xi}$, $N, k \in \mathbb{Z}_+$ and $k' \in \mathbb{Z}$, where $F(0) = 1$. If $F(\pi) \neq -1$ and the coefficients of $F(\xi)$ satisfy

$$a[k] \equiv \left(\sum_{k;even\ j} |a_j|\right)^2 + \left(\sum_{k;odd\ j} |a_j|\right)^2 < 2^{2N-1}, \tag{2.1}$$

then ϕ is in $L_2(\mathbb{R})$.

Remark 2.1 Consider the example shown in [2] (see PP. 542-544 in [2]), we find that the function $\phi = \Pi_{j=1}^{\infty} m_0^N(2^{-j}\xi)$ with the mask

$$m_0^1(\xi) = e^{i\xi}\left(\frac{1+e^{-i\xi}}{2}\right)\left(-\frac{1}{2} + 2e^{-i\xi} - \frac{1}{2}e^{-2i\xi}\right)$$

is not in $L_2(\mathbb{R})$.

It is easy to check that $m_0^1(\xi)$ satisfies

$$\left(\sum_{k;even\ j} |a_j|\right)^2 + \left(\sum_{k;odd\ j} |a_j|\right)^2 = 5 = 2^{2N-1} + 3,$$

where $N = 1$. This example shows that a necessary condition for $\phi \in \Phi$ being square integrable is

$$\left(\sum_{k;even\ j} |a_j|\right)^2 + \left(\sum_{k;odd\ j} |a_j|\right)^2 \geq 2^{2N-1} + 3,$$

where Φ is the set defined in Definition 1.1.

Condition (2.1) can be replaced by stronger conditions,

$$\sum_{j=0}^{k} |a_j| < 2^{N-\frac{1}{2}},$$

due to the obvious inequality $a^2 + b^2 \leq (a+b)^2$ for all $a, b \geq 0$.

Proof. It is sufficient to prove the boundedness of the following integral

$$\begin{aligned}
&\int_{|\xi|\geq\pi}|\hat{\phi}(\xi)|^2d\xi\\
=&\sum_{\ell=1}^{\infty}\int_{2^{\ell-1}\pi\leq|\xi|\leq2^{\ell}\pi}\Pi_{j=1}^{\infty}\left|\left(\frac{1+e^{-i2^{-j}\xi}}{2}\right)^N F(2^{-j}\xi)\right|^2 d\xi\\
=&\sum_{\ell=1}^{\infty}\int_{2^{\ell-1}\pi\leq|\xi|\leq2^{\ell}\pi}\left|\frac{1-e^{-i\xi}}{i\xi}\right|^{2N}\Pi_{j=1}^{\infty}|F(2^{-j}\xi)|^2d\xi\\
\leq& C\sum_{\ell=1}^{\infty}\int_{2^{\ell-1}\pi\leq|\xi|\leq2^{\ell}\pi}\frac{1}{|\xi|^{2N}}\Pi_{j=1}^{\infty}|F(2^{-j}\xi)|^2d\xi\\
\leq& C\sum_{\ell=1}^{\infty}\frac{1}{2^{2\ell N}}\int_{2^{\ell-1}\pi\leq|\xi|\leq2^{\ell}\pi}\Pi_{j=1}^{\ell}|F(2^{-j}\xi)|^2\Pi_{j=1}^{\infty}|F(2^{-\ell-j}\xi)|^2d\xi\\
\leq& C\sum_{\ell=1}^{\infty}4^{-\ell N}\int_{2^{\ell-1}\pi\leq|\xi|\leq2^{\ell}\pi}\Pi_{j=1}^{\ell}|F(2^{-j}\xi)|^2d\xi. \qquad (2.2)
\end{aligned}$$

We now prove the boundedness of the last integral in inequality (2.2). Denote

$$Tf(\xi)=\left|F\left(\frac{\xi}{2}\right)\right|^2 f\left(\frac{\xi}{2}\right)+\left|F\left(\frac{\xi}{2}+\pi\right)\right|^2 f\left(\frac{\xi}{2}+\pi\right).$$

Hence, for any 2π-periodic continuous function f, we have

$$\begin{aligned}
&\int_{2^{\ell-1}\pi\leq|\xi|\leq2^{\ell}\pi}f(2^{-\ell}\xi)\Pi_{j=1}^{\ell}\left|F(2^{-j}\xi)\right|^2 d\xi\\
=&\int_{-\pi}^{\pi}T^{\ell}f(\xi)d\xi\leq\sqrt{2\pi}||T^{\ell}f||_{L^2}\leq\sqrt{2\pi}||f||_{L^2}||T^{\ell}||.
\end{aligned}$$

Let $\rho(T)$ be the spectral radius of the opertor T. Since $F(0)=1$ and $F(\pi)\neq-1$, it can be shown that $\rho(T)>0$ (see also [1]). For every $\epsilon>0$, there is an integer $\ell(\epsilon)$ such that

$$||T^{\ell}||\leq\left(\rho(T)+\epsilon\right)^{\ell},\qquad \ell>\ell(\epsilon).$$

It follows from (2.2) that

$$\int_{|\xi|\geq\pi} |\hat{\phi}(\xi)|^2 d\xi \leq C \sum_{\ell=1}^{\ell(\epsilon)} 4^{-N\ell} ||T^\ell|| + C \sum_{\ell=\ell(\epsilon)+1}^{\infty} 4^{-N\ell} \left(\rho(T)+\epsilon\right)^\ell ;$$

so $\rho(T)$ must be estimated if we are to choose an $\epsilon > 0$ small enough for the series to converge. Regardless of how small an $\epsilon > 0$ is chosen, the contribution

$$C \sum_{\ell=1}^{\ell(\epsilon)} 4^{-N\ell} ||T^\ell|| \leq C \sum_{\ell=1}^{\ell(\epsilon)} 4^{-N\ell} ||T||^\ell$$

is finite, although possibly large.

To evaluate $\rho(T)$, we consider the conjugate operator, T^*, of T. It is easy to find that

$$T^* f(\xi) = 2\left|F(\xi)\right|^2 f(2\xi).$$

In fact, for any 2π-periodic continuous functions f and g,

$$\begin{aligned}
& \langle Tf, g\rangle \\
= & \int_{-\pi}^{\pi} |F\left(\frac{\xi}{2}\right)|^2 f\left(\frac{\xi}{2}\right) \bar{g}(\xi) d\xi + \int_{-\pi}^{\pi} |F\left(\frac{\xi}{2}+\pi\right)|^2 f\left(\frac{\xi}{2}+\pi\right) \bar{g}(\xi) d\xi \\
= & 2\left(\int_{-\pi/2}^{\pi/2} |F\left(\xi\right)|^2 f\left(\xi\right) \bar{g}(2\xi) d\xi + \int_{\pi/2}^{3\pi/2} |F\left(\xi\right)|^2 f\left(\xi\right) \bar{g}(2\xi) d\xi\right) \\
= & 2\int_{-\pi/2}^{3\pi/2} |F\left(\xi\right)|^2 f\left(\xi\right) \bar{g}(2\xi) d\xi \\
= & 2\int_{-\pi}^{\pi} |F\left(\xi\right)|^2 f\left(\xi\right) \bar{g}(2\xi) d\xi = \langle f, T^* g\rangle.
\end{aligned}$$

If we consider the Fourier expression

$$|F(\xi)|^2 = \sum_{\ell=-k}^{k} b_\ell e^{i\ell\xi},$$

then the matrix of T^* restricted to $E_k = \{\sum_{\ell=-k}^{k} c_\ell e^{i\ell\xi}, (c_{-k}, \cdots, c_k) \in C^{2k+1}\}$ is given by

$$2H = 2\left(b_{j-2i}\right)_{i,j=-k,\cdots,k}.$$

It is clear that b_t can be written as

$$b_t = \sum_{j=0}^{k-|t|} a_{k-|t|-j}a_{k-j}, \quad t = -k, \cdots, k. \tag{2.3}$$

Hence, $b_t = b_{-t}$, for all $t = -k, \cdots, k$. It is also obvious that b_k is an eigenvalue of H with multiplicity 2. To estimate bounds of the eigenvalues of H, we consider the maximum column sum matrix norm $\|\cdot\|_1$ of $H \equiv H_k$ for $k = 2m$ and $2m+1$:

$$\|H_{2m}\|_1 = \max_{-k\le j\le k} \sum_{i=-k}^{k} |b_{j-2i}| = \max\left\{ \sum_{t=-m}^{m} |b_{2t}|, \sum_{t=-m}^{m-1} |b_{2t+1}| \right\}, \tag{2.4}$$

where

$$\begin{aligned} \sum_{t=-m}^{m} |b_{2t}| &= 2\sum_{t=1}^{m} |b_{2t}| + b_0 \\ &= 2\sum_{t=1}^{m} \left| \sum_{j=0}^{2m-2t} a_{2m-2t-j}a_{2m-j} \right| + b_0 \\ &\le 2\sum_{t=1}^{m} \sum_{j=0}^{2m-2t} |a_{2m-2t-j}a_{2m-j}| + b_0. \end{aligned} \tag{2.5}$$

In (2.5) we substitute $\ell = 2m - j$ and obtain

$$\begin{aligned}
\sum_{t=-m}^{m} |b_{2t}| &\leq 2\sum_{t=1}^{m}\sum_{\ell=2t}^{2m} |a_{\ell-2t}a_{\ell}| + \sum_{t=0}^{2m} a_t^2 \\
&= 2\sum_{t=1}^{m}\sum_{u=t}^{m} |a_{2u-2t}a_{2u}| + 2\sum_{t=1}^{m}\sum_{u=t}^{m-1} |a_{2u-2t+1}a_{2u+1}| \\
&\quad + \sum_{t=0}^{m} a_{2t}^2 + \sum_{j=0}^{m-1} a_{2t+1}^2 \\
&= \sum_{t=0}^{m}\sum_{u=t}^{m} |a_{2u-2t}a_{2u}| + \sum_{t=1}^{m}\sum_{u=t}^{m} |a_{2u-2t}a_{2u}| \\
&\quad + \sum_{t=0}^{m-1}\sum_{u=t}^{m-1} |a_{2u-2t+1}a_{2u+1}| + \sum_{t=1}^{m-1}\sum_{u=t}^{m-1} |a_{2u-2t+1}a_{2u+1}| \\
&= \sum_{u=0}^{m}\sum_{t=0}^{u} |a_{2u-2t}a_{2u}| + \sum_{u=1}^{m}\sum_{t=1}^{u} |a_{2u-2t}a_{2u}| \\
&+ \sum_{u=0}^{m-1}\sum_{t=0}^{u} |a_{2u-2t+1}a_{2u+1}| + \sum_{u=1}^{m-1}\sum_{t=1}^{u} |a_{2u-2t+1}a_{2u+1}| \,. \quad (2.6)
\end{aligned}$$

In the rightmost equality of (2.6), substituting $u - t = j$ yields

$$\begin{aligned}
\sum_{t=-m}^{m} |b_{2t}| &\leq \sum_{u=0}^{m}\sum_{j=0}^{u} |a_{2j}||a_{2u}| + \sum_{u=1}^{m}\sum_{j=0}^{u-1} |a_{2j}||a_{2u}| \\
&\quad + \sum_{u=0}^{m-1}\sum_{j=0}^{u} |a_{2j+1}||a_{2u+1}| + \sum_{u=1}^{m-1}\sum_{j=0}^{u-1} |a_{2j+1}||a_{2u+1}|.
\end{aligned}$$

Then, we switch the two sums of the second summation and the fourth summation on the right-hand side of the above equality and combine the new second summation with the first summation and the new fourth summation with the third summation. Thus, we obtain

$$\begin{aligned}
\sum_{t=-m}^{m}|b_{2t}| &\le \sum_{u=0}^{m}\sum_{j=0}^{u}|a_{2u}||a_{2j}|+\sum_{j=0}^{m-1}\sum_{u=j+1}^{m}|a_{2u}||a_{2j}| \\
&\quad+\sum_{u=0}^{m-1}\sum_{j=0}^{u}|a_{2u+1}||a_{2j+1}|+\sum_{j=0}^{m-2}\sum_{u=j+1}^{m-1}|a_{2u+1}||a_{2j+1}| \\
&= \sum_{u=0}^{m}\sum_{j=0}^{u}|a_{2u}||a_{2j}|+\sum_{u=0}^{m-1}\sum_{j=u+1}^{m}|a_{2u}||a_{2j}| \\
&\quad+\sum_{u=0}^{m-1}\sum_{j=0}^{u}|a_{2u+1}||a_{2j+1}|+\sum_{u=0}^{m-2}\sum_{j=u+1}^{m-1}|a_{2u+1}||a_{2j+1}| \\
&= \sum_{u=0}^{m}\sum_{j=0}^{m}|a_{2u}||a_{2j}|+\sum_{u=0}^{m-1}\sum_{j=0}^{m-1}|a_{2u+1}||a_{2j+1}| \\
&= \left(\sum_{j=0}^{m}|a_{2j}|\right)^2+\left(\sum_{j=0}^{m-1}|a_{2j+1}|\right)^2. \qquad (2.7)
\end{aligned}$$

Similarly, we have

$$\sum_{t=-m}^{m-1}|b_{2t+1}|\le\left(\sum_{j=0}^{m}|a_{2j}|\right)^2+\left(\sum_{j=0}^{m-1}|a_{2j+1}|\right)^2. \qquad (2.8)$$

From (2.4), (2.7) and (2.8) we find the spectral radius of H_{2m} as

$$\rho(H_{2m})\le\|H_{2m}\|_1\le\left(\sum_{j=0}^{m}|a_{2j}|\right)^2+\left(\sum_{j=0}^{m-1}|a_{2j+1}|\right)^2.$$

By using the same argument, we find that

$$\rho(H_{2m+1})\le\|H_{2m+1}\|_1\le\left(\sum_{j=0}^{m}|a_{2j}|\right)^2+\left(\sum_{j=0}^{m-1}|a_{2j+1}|\right)^2.$$

It follows that

$$\rho(T)=\rho(T^*)=2\rho(H)\le 2\left(\sum_{j=0}^{m}|a_{2j}|\right)^2+2\left(\sum_{j=0}^{m-1}|a_{2j+1}|\right)^2.$$

If (2.1) holds, i.e.,

$$\left(\sum_{j=0}^{m}|a_{2j}|\right)^2+\left(\sum_{j=0}^{m-1}|a_{2j+1}|\right)^2\le 2^{2N-1},$$

then $\rho(T)<2^{2N}$. So we choose

$$\epsilon=\frac{1}{2}\left(2^{2N}-\rho(T)\right).$$

Therefore

$$\rho(T)+\epsilon<2^{2N},$$

and we obtain the estimation

$$\int_{|\xi|\ge\pi}\left|\hat{\phi}(\xi)\right|^2 d\xi\le C\sum_{\ell=1}^{\ell(\epsilon)}4^{-N\ell}||T||^{\ell}+\sum_{\ell=\ell(\epsilon)+1}^{\infty}\left(\frac{\rho(T)+\epsilon}{4^N}\right)^{\ell}.$$

The tail of the series is a convergent geometric series, thus completing the proof of the theorem.

By using Theorem 2.1 and noting Remark 2.1, we immediately have the following improvement of Theorem 2 in [6].

Theorem 2.2 *Let $\phi=\Pi_{j=1}^{\infty}m_0^N(2^{-j}\xi)$ and $\tilde{\phi}=\Pi_{j=1}^{\infty}\tilde{m}_0^{\tilde{N}}(2^{-j}\xi)$ be two B-spline type scaling functions defined by 1.1, and let $m_0^N(\xi)$ and $\tilde{m}_0^{\tilde{N}}(\xi)$ be in the form of (1.6) and (1.7), respectively. If*

(i)

$$a[k]\equiv\left(\sum_{k;even\ j}|a_j|\right)^2+\left(\sum_{k;odd\ j}|a_j|\right)^2<2^{2N-1},$$
$$\tilde{a}[\tilde{k}]\equiv\left(\sum_{\tilde{k};even\ j}|\tilde{a}_j|\right)^2+\left(\sum_{\tilde{k};odd\ j}|\tilde{a}_j|\right)^2<2^{2\tilde{N}-1},\ and$$

(ii) $\sum_{j=\mu}^{\nu}\sum_{\ell=0}^{k}\sum_{\tilde{\ell}=0}^{\tilde{k}}\binom{\tilde{N}}{j-\tilde{\ell}-\tilde{k}'}\binom{N}{j+2n-\ell-k'}\tilde{a}_{\tilde{\ell}}a_{\ell}=2^{N+\tilde{N}-1}\delta_{n0},$

where $\mu = min\{k', \tilde{k}'\}$; $\nu = max\{N + k + k', \tilde{N} + \tilde{k} + \tilde{k}'\}$; δ_{n0} *is the Kronecker symbol; and* $n = 0, \pm 1, \pm 2, \cdots$. *Then we have* $\phi, \tilde{\phi} \in L^2(R)$ *and* $\langle \phi(t), \tilde{\phi}(t-i) \rangle = \delta_{i,0}$ *for all* $i \in Z$. *The corresponding* ψ *and* $\tilde{\psi}$ *define biorthogonal wavelets (biorthogonal Riesz bases) and are in* C^{α} *and* $C^{\tilde{\alpha}}$, *respectively. Here* α *and* $\tilde{\alpha}$ *are more than*

$$N - \frac{1}{2}\log_2(2a[k]) \text{ and } \tilde{N} - \frac{1}{2}\log_2\left(2\tilde{a}[\tilde{k}]\right),$$

respectively.

Proof. By using Theorem 2.1 and arguments similar to that in the proofs of Lemmas 4, 5, and 7 in [6], we can complete the proof of the theorem.

The algorithm shown in [6] for constructing biorthogonal scaling functions ϕ and $\tilde{\phi}$ with the largest possible regularity and the required vanishing moments can be improved by the following optimization problem of finding suitable $F(\xi)$ and $\tilde{F}(\xi)$, or, equivalently, suitable coefficient sets, $\mathbf{a} = \{a_0, \cdots, a_k\}$ and $\tilde{\mathbf{a}} = \{\tilde{a}_0, \cdots, \tilde{a}_k\}$, of $F(\xi)$ and $\tilde{F}(\xi)$, respectively, such that $a[k]$ and $\tilde{a}[\tilde{k}]$ are the minimum under conditions (i) and (ii) of Theorem 2.2. For wavelet analysis of spline approximation, we usually assume $F(\xi) = 1$; i.e., the corresponding ϕ is the B-spline of order N. Consequently, the optimization problem can be written as follows.

$$\min_{\tilde{\mathbf{a}}} \qquad \tilde{a}[\tilde{k}] \equiv \left(\sum_{\tilde{k};even\ j} |\tilde{a}_j|\right)^2 + \left(\sum_{\tilde{k};odd\ j} |\tilde{a}_j|\right)^2, \tag{2.9}$$

$$\text{subject to} \qquad \left(\sum_{j=0}^{\tilde{k}} (-1)^j \tilde{a}_j + 1\right)^2 > 0, \tag{2.10}$$

$$\tilde{a}[\tilde{k}] < 2^{2\tilde{N}-1}, \tag{2.11}$$

$$\sum_{\ell=0}^{\tilde{k}} \sum_{j=\tilde{k}'}^{\max\{N,\tilde{N}+\tilde{k}+\tilde{k}'\}} \binom{\tilde{N}}{j-\ell-\tilde{k}'} \binom{N}{j+2n} \tilde{a}_\ell = 2^{N+\tilde{N}-1}\delta_{n0}, \tag{2.12}$$

where object (2.9) will give the largest possible regularity, condition (2.10) is from the definition of $\tilde{F}$, and conditions (2.11) and (2.12) come from conditions (i) and (ii) of Theorem 2.2.

As examples, we consider $m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)$; i.e., ϕ is the B-spline of order 1. If we choose $\tilde{N} = 1$ and $\tilde{k} = \tilde{k}' = 0$, then the solution of problem (2.9)-(2.12) is $\tilde{a}_0 = 1$ and we arrive at the Haar function. If we choose $\tilde{N} = 2$, $\tilde{k} = 1$, and $\tilde{k}' = 0$, then the solutions are $\tilde{a}_0 = 3/2$ and $\tilde{a}_1 = -1/2$. Hence, the corresponding $\tilde{\phi}$ is defined by $\hat{\tilde{\phi}}(\xi) = \Pi_{j=1}^{\infty} \tilde{m}_0^2(2^{-j}\xi)$, where $\tilde{m}_0^2(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^2 \left(\frac{3-e^{-i\xi}}{2}\right)$. The regularities of ϕ and $\tilde{\phi}$ are more than 0.5 and $2 - \log_2(5)/2 = 0.839036$, respectively.

If we choose $N = 2$, $\tilde{N} = 2$, $\tilde{k} = 2$, and $\tilde{k}' = -1$, then the solutions of the optimization problem are $\tilde{a}_0 = -1/2$, $\tilde{a}_2 = 2$, and $\tilde{a}_3 = -1/2$. Hence, the corresponding ϕ is the B-spline of order 2, and $\tilde{\phi}$ is defined by $\hat{\tilde{\phi}}(\xi) = \Pi_{j=1}^{\infty} \tilde{m}_0^2(2^{-j}\xi)$, where $\tilde{m}_0^2(\xi) = e^{i\xi} \left(\frac{1+e^{-i\xi}}{2}\right)^2 \left(-\frac{1}{2} + 2e^{-i\xi} - \frac{1}{2}e^{-2i\xi}\right)$. The regularities of ϕ and $\tilde{\phi}$ are more than 1.5 and $2 - \log_2(5)/2 = 0.339036$, respectively.

3 Construction of sequences of biorthogonal B-spline type scaling functions

In this section, we will give a a method for constructing a sequence of B-spline type scaling functions and wavelets from either an orthogonal sacaling function or a pair of biorthogonal scaling functions by using their convolutions with certain B-splines.

Denote $B_n(t)$ the B-spline of degree $n-1$ having nodes at $0, 1, \cdots, n$; i.e., $B_n(t) = Q_n(t) = M(t; 0, 1, \cdots, n)$, which is defined by (1.1) on page 11 of Schoenberg [12]. The Fourier transform of $B_n(t)$ is

$$\widehat{B_n}(\xi) = e^{-in\xi/2} \left(\frac{\sin(\xi/2)}{\xi/2}\right)^n = \left(\frac{1 - e^{-i\xi}}{i\xi}\right)^n. \tag{3.1}$$

Another type of B-spline functions we need are denoted by $C_n(t)$ ($n = 1, 2, \cdots$) that are defined as

$$C_n(t) = M(t; -n, \cdots, -1, 0), \tag{3.2}$$

where function M is given in (1.1) of [12]. Therefore, the Fourier transform of $C_n(t)$ is

$$\widehat{C_n}(\xi) = e^{in\xi/2}\left(\frac{\sin(\xi/2)}{\xi/2}\right)^n. \tag{3.3}$$

Comparing the above expression with Eq. (3.1) yields

$$\overline{\widehat{C_n}}(\xi) = \widehat{B_n}(\xi). \tag{3.4}$$

Eq. (3.4) can also be derived from the symmetry of B-splines. The symmetry $C_n(t) = B_n(-t)$ leads to the Fourier transform of $\widehat{C_n(t)}(\xi) = \widehat{B_n(-t)}(\xi) = \overline{\widehat{B_n(t)}}(\xi)$.

In this section, we denote by $\phi(t)$ an orthogonal scaling function that satisfies the dilation equation (or refinement equation)

$$\phi(t) = \sum_k c_k \phi(2t-k), \tag{3.5}$$

where the constant coefficients c_k satisfy the following four properties.

(i) $c_k = 0$ for $k \notin \{0, 1, \cdots, 2p-1\}$;

(ii) $\sum_k c_k = 2$;

(iii) $\sum_k (-1)^k k^m c_k = 0$ for $0 \le m \le p-1$;

(iv) $\sum_k c_k c_{k-2m} = 2\delta_{0m}$ for $1-p \le m \le p-1$.

If ϕ and $\tilde{\phi}$ are biorthogonal scaling functions with refinement expressions

$$\phi(t) = \sum_n c_n \phi(2t-n), \quad \tilde{\phi}(t) = \sum_n \tilde{c}_n \phi(2t-n), \tag{3.6}$$

then the coefficients c_k and $\tilde{c}_k$ satisfy

(i)' $c_k = \tilde{c}_{\tilde{k}} = 0$ for $k \notin \{0, 1, \cdots, 2p-1\}$ and $\tilde{k} \notin \{0, 1, \cdots, 2\tilde{p}-1\}$;

(ii)' $\sum_k c_k = \sum_k \tilde{c}_k = 2$;

(iii)' $\sum_k (-1)^k k^m c_k = \sum_k (-1)^k k^{\tilde{m}} \tilde{c}_k = 0$ for $0 \le m \le p-1$ and $0 \le \tilde{m} \le \tilde{p}-1$;

(iv)' $\sum_k c_k \tilde{c}_{k-2m} = 2\delta_{0m}$ for $1-\widetilde{p} \le m \le \widetilde{p}-1$.

Biorthogonal B-spline Wavelet Sequences

Let $\phi(t) = \sum_{k=0}^{2p-1} c_k \phi(2t-k)$ and $\phi(t) \in L^2(\mathbb{R})$. We define $\overline{\phi}_n := \phi * B_n$. Clearly ϕ_n is in $L^2(\mathbb{R})$. We also have the following results on ϕ_n.

Theorem 3.1 *Let $\phi(t) = \sum_{k=0}^{2p-1} c_k \phi(2t-k)$, and let $\phi(t) \in L^2(\mathbb{R})$ satisfy (i)-(iv). Then $\overline{\phi}_n := \phi * B_n$ satisfies the dilation equation*

$$\overline{\phi}_n = \sum_{k=0}^{2p+n-1} h_k^{(n)} \overline{\phi}_n(2t-k), \tag{3.7}$$

where

$$h_k^{(n)} = \frac{1}{2^n} \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) c_{k-j}. \tag{3.8}$$

In addition, $h^{(n)}$ as shown in (3.8) possesses the following properties.

(i)" $h_k^{(n)} = 0$ *for* $k \notin \{0, 1, \cdots, 2p+n-1\}$;

(ii)" $\sum_k h_k^{(n)} = 2$;

(iii)" $\sum_k (-1)^k k^m h_k^{(n)} = 0$ *for* $0 \le m \le p+n-1$.

Proof. Denote the mask of ϕ by $m_\phi(\xi)$. Then from (3.1), the mask of $\overline{\phi}_n$ is

$$m_{\overline{\phi}_n}(\xi) = \left(\frac{1+e^{-i\xi}}{2} \right)^n m_\phi(\xi).$$

Consequently,

$$\begin{aligned} m_{\overline{\phi}_n}(\xi) &= \frac{1}{2^n} \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) e^{-ij\xi} \frac{1}{2} \sum_{k=0}^{2p-1} c_k e^{-ik\xi} \\ &= \frac{1}{2} \sum_{k=0}^{2p+n-1} \left(\frac{1}{2^n} \sum_{j=0}^{k} \left(\begin{array}{c} n \\ j \end{array} \right) c_{k-j} \right) e^{-ik\xi}. \end{aligned}$$

Noting $\left(\begin{array}{c} n \\ j \end{array} \right) = 0$ for $j > n$, we have Eqs. (3.7) and (3.8).

From items (i) and (3.8), we immediately obtain (i)". Item (ii)" also holds because

$$\sum_{k=0}^{2p+n-1} h_k^{(n)} = \sum_{k=0}^{2p+n-1} \frac{1}{2^n} \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) c_{k-j}$$

$$= \sum_{j=0}^{n} \frac{1}{2^n} \left(\begin{array}{c} n \\ j \end{array} \right) \sum_{k'=-j}^{2p+n-j-1} c_{k'} = \sum_{j=0}^{n} \frac{1}{2^n} \left(\begin{array}{c} n \\ j \end{array} \right) \sum_{k'=0}^{2p-1} c_{k'} = 2.$$

Finally, we prove item (iii)". For $m = 0, 1, \cdots, p+n-1$,

$$\begin{aligned}
&\sum_{k=0}^{2p+n-1} (-1)^k k^m h_k^{(n)} \\
&= \frac{1}{2^n} \sum_{k=0}^{2p+n-1} \sum_{j=0}^{n} (-1)^k k^m \left(\begin{array}{c} n \\ j \end{array} \right) c_{k-j} \\
&= \frac{1}{2^n} \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) \sum_{k=-j}^{2p+n-j-1} (-1)^{k+j} (k+j)^m c_k \\
&= \frac{1}{2^n} \sum_{j=0}^{n} (-1)^j \left(\begin{array}{c} n \\ j \end{array} \right) \sum_{k=0}^{2p-1} (-1)^k c_k \sum_{i=0}^{m} \left(\begin{array}{c} m \\ i \end{array} \right) k^i j^{m-i} \\
&= \frac{1}{2^n} \sum_{i=0}^{m} \left(\begin{array}{c} m \\ i \end{array} \right) \left(\sum_{j=0}^{n} (-1)^j \left(\begin{array}{c} n \\ j \end{array} \right) j^{m-i} \right) \left(\sum_{k=0}^{2p-1} (-1)^k k^i c_k \right).
\end{aligned}$$

Hence, for $0 \leq i \leq p-1$, the sum in the second parenthesis in the last line of the above equation is equal to zero (see item (iii)). For $p \leq i \leq m$, we have $0 \leq m-i \leq m-p \leq p+n-1-p = n-1$. Therefore, from the identity

$$\sum_{j=0}^{n} (-1)^j \left(\begin{array}{c} n \\ j \end{array} \right) j^\ell = (-1)^n n! \delta_{\ell n}$$

($\ell = 0, 1, \cdots, n$), the sum in the first parenthesis in the last line of the above equation becomes zero. It follows that the right-hand side of the last equation is equal to zero. This completes the proof of theorem.

From Theorem 3.1, we know that $\overline{\phi}_n$ is a scaling function with the approximation degree $p+n-1$. However, it does not satisfy orthogonal

condition (iv). Thus, we need to define a biorthogonal scaling function, $\widetilde{\phi}_n$, with respect to $\overline{\phi}_n$ by

$$\left(C_n * \widetilde{\phi}_n\right)(t) = \phi(t). \tag{3.9}$$

Taking Fourier transforms on both sides of Eq. (3.9) while noting Eq. (3.3) leads to

$$\widehat{\widetilde{\phi}_n}(\xi) = \frac{\widehat{\phi}(\xi)}{\widehat{C_n}(\xi)} = e^{-in\xi/2}\left(\frac{\xi/2}{\sin(\xi/2)}\right)^n \widehat{\phi}(\xi). \tag{3.10}$$

Theorem 3.2 *The function $\widetilde{\phi}_n$ is well defined by Eq. (3.9), and it satisfies the following dilation equation.*

$$\widehat{\widetilde{\phi}_n}(\xi) = \widetilde{m}_{\widetilde{\phi}_n}\left(\frac{\xi}{2}\right)\widehat{\widetilde{\phi}_n}\left(\frac{\xi}{2}\right), \tag{3.11}$$

where

$$\widetilde{m}_{\widetilde{\phi}_n}(\xi) = \left(\frac{2}{1+e^{i\xi}}\right)^n m_\phi(\xi)$$

and m_ϕ is the mask of ϕ.

In addition, let $\overline{\phi}_n$ be defined as in Theorem 3.1. Then $\{\overline{\phi}_n(t-k)\}_{k\in\mathbb{Z}}$ and $\{\widetilde{\phi}_n(t-k)\}_{k\in\mathbb{Z}}$ are biorthogonal sets; i.e., $\overline{\phi}_n$ and $\widetilde{\phi}_n$ satisfy $\langle\overline{\phi}_n(t), \widetilde{\phi}_n(t-k)\rangle = \delta_{0k}$.

Proof. To derive dilation equation (3.11), we start from Eq. (3.10) and apply Eqs. (3.4) and (3.9) to give

$$\begin{aligned}\frac{\widehat{\widetilde{\phi}_n}(\xi)}{\widehat{\widetilde{\phi}_n}(\xi/2)} &= \frac{\widehat{C_n}(\xi/2)}{\widehat{C_n}(\xi)}\frac{\phi(\xi)}{\phi(\xi/2)} = \frac{\overline{\widehat{B_n}(\xi/2)}}{\widehat{B_n}(\xi)} m_\phi\left(\frac{\xi}{2}\right) \\ &= \left(\frac{2}{1+e^{i\xi}}\right)^n m_\phi\left(\frac{\xi}{2}\right).\end{aligned}$$

Hence, Eq. (3.11) is established and $\widetilde{\phi}_n$ is well defined by (3.9).

To prove the biorthogonality of $\{\overline{\phi}_n(t-k)\}_{k\in\mathbb{Z}}$ and $\{\widetilde{\phi}_n(t-k)\}_{k\in\mathbb{Z}}$, we use the General Parseval's Relation, substituting expressions shown as in (3.1) and (3.10), and noting relation (3.4), we have

$$\begin{aligned}
&\langle\overline{\phi}_n(t), \widetilde{\phi}_n(t-k)\rangle = \frac{1}{2\pi}\langle\widehat{\overline{\phi}_n}(\xi), \widehat{\widetilde{\phi}_n}(\xi)e^{-ik\xi}\rangle \\
= &\frac{1}{2\pi}e^{ik\xi}\langle\widehat{B_n}(\xi)\widehat{\phi}(\xi), \frac{\widehat{\phi}(\xi)}{\widehat{C_n}(\xi)}\rangle = \frac{1}{2\pi}e^{ik\xi}\langle\widehat{\phi}(\xi), \widehat{\phi}(\xi)\rangle \\
= &\langle\phi(t), \phi(t-k)\rangle = \delta_{ok},
\end{aligned}$$

where the last step is due to the orthonormality of $\{\phi(t-k)\}_{k\in\mathbb{Z}}$.

Similar to Theorem 3.1, we derive the following properties of the dilation coefficients of $\widetilde{\phi}_n$.

Theorem 3.3 *Let $\phi(t) = \sum_{k=0}^{2p-1} c_k\phi(2t-k)$. Then $\widetilde{\phi}_n$, $n \le 2p-1$, defined as in (3.9) satisfies the dilation equation*

$$\widetilde{\phi}_n = \sum_{k=n}^{2p-1} \widetilde{h}_k^{(n)}\widetilde{\phi}_n(2t-k), \tag{3.12}$$

where dilation coefficients, $\{\widetilde{h}_k^{(n)}\}_k$, satisfy

$$c_k = \frac{1}{2^n}\sum_{j=0}^{n}\begin{pmatrix} n \\ j \end{pmatrix}\widetilde{h}_{k+j}^{(n)}$$

for $k = n, n+1, \cdots, 2p-1$.

More properties on $\widetilde{h}_k^{(n)}$ $(n \le k \le 2p-1)$, the dilation coefficients of $\widetilde{\phi}_n$, and computation of the coefficients will be discussed in a later paper. In the following we shall determine the classes the functions $\widetilde{\phi}_n$ $(1 \le n \le 2p-1)$ may be in which.

Although $\phi \in L^2(\mathbb{R})$ leads to $\overline{\phi}_n = B_n * \phi \in L^2(\mathbb{R})$, it cannot guarantee that $\widetilde{\phi}_n$ is also in $L^2(\mathbb{R})$. However, if $\widetilde{\phi}_n$ is the distribution solution to the dilation equation (3.12), then $\widehat{\widetilde{\phi}_n}(0) = 1$, and from [14], $\widetilde{\phi}_n \in H^{-s}$ for $s > \log_2\sum_k \left|\widetilde{h}_k^{(n)}\right| - 1/2$. The proof follows from the fact that

$$\left|\widehat{\widetilde{\phi}_n}(\xi)\right| \le C(1+|\xi|)^M,$$

where $M = \log_2\sum_k \left|h_k^{(n)}\right| - 1$, and hence $\widetilde{\phi}_n \in H^{-s}$ for $s > M + 1/2$. Therefore, scaling function $\widetilde{\phi}_n$ generates a multiresolution analysis of H^{-s}.

Using Theorem 2.1, we now give a condition for the orthogonal scaling function, $\phi \in \Phi$, defined in Definition 1.1 such that its corresponding $\widetilde{\phi}_n$ is in $L^2(\mathbb{R})$.

Theorem 3.4 *Let ϕ be defined as*

$$\hat{\phi} = \prod_{j=1}^{\infty} m_0^N(2^{-j}\xi),$$

where $m_0^N(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N F(\xi)$ with $F(\xi) = e^{-ik'\xi}\sum_{j=0}^{k} a_j e^{-ij\xi}$ is defined by (1.3) and (1.4). If $F(\pi) \neq -1$ and the coefficients of $F(\xi)$ satisfy

$$\left(\sum_{k;even\ j} |a_j|\right)^2 + \left(\sum_{k;odd\ j} |a_j|\right)^2 < 2^{2N-1}, \tag{3.13}$$

then ϕ is in $L_2(\mathbb{R})$, while both ϕ and $\widetilde{\phi}_n$ $(n < N)$ are in $L^2(\mathbb{R})$ when

$$\left(\sum_{k;even\ j} |a_j|\right)^2 + \left(\sum_{k;odd\ j} |a_j|\right)^2 < 2^{2(N-n)-1}. \tag{3.14}$$

We now extend the results for the orthogonal scaling functions to the the biorthogonal scaling functions.

Theorem 3.5 *Let $f(t) = \sum_{k=0}^{2p-1} c_k f(2t-k)$ and $g(t) = \sum_{k=0}^{2\widetilde{p}-1} d_k g(2t-k)$ be biorthogonal scaling functions (i.e., $\langle f(t), g(t-\ell)\rangle = \delta_{0\ell}$) with approximation degrees p and $\widetilde{p}$, respectively. Then functions $\overline{f}_n$ and $\widetilde{g}_n$ defined by*

$$\overline{f}_n(t) = (B_n * f)(t), \text{ and } (C_n * \widetilde{g}_n)(t) = g(t), \tag{3.15}$$

where B_n and C_n are B-splines, are also biorthogonal scaling functions with approximation degrees $p+n-1$ and $\widetilde{p}-n-1$, respectively.

In addition, if $f \in L^2(\mathbb{R})$, then $\overline{f}_n$ is also in $L^2(\mathbb{R})$. If the Fourier transform of g can be written as

$$\widehat{g}(\xi) = \prod_{j=1}^{\infty} m_0^N(2^{-j}\xi),$$

where $m_0^N(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N F(\xi)$ *with* $F(\xi) = e^{-ik'\xi}\sum_{j=0}^{k} a_j e^{-ij\xi}$ *is defined by (1.3) and (1.4), with* $F(\pi) \neq -1$ *and the coefficients of* $F(\xi)$ *satisfy*

$$\left(\sum_{k;even\ j} |a_j|\right)^2 + \left(\sum_{k;odd\ j} |a_j|\right)^2 < 2^{2(N-n)-1},$$

then both g *and* $\widetilde{g}_n$ *are also in* $L^2(\mathbb{R})$.

Example 3.1. Considering the examples given in [2] and [4] (or see the examples shown at the end of Section 2), let ${}_N\phi$ and ${}_{N,\widetilde{N}}\widetilde{\phi}$ be a pair of biorthogonal scaling functions with masks ${}_Nm_0$ and ${}_{N,\widetilde{N}}\widetilde{m}_0$, respectively. It is easy to check that $\overline{{}_N\phi}_n(t) =_{N+n} \phi(t-k)$ and $\widetilde{\widetilde{{}_{N,\widetilde{N}}\phi}}_n(t) =_{N+n,\widetilde{N}-n} \widetilde{\phi}(t-\ell)$ $(n < \widetilde{N})$ for some integers k and ℓ, where $\overline{f}_n$ and $\widetilde{g}_n$ are defined by Eqs. in (3.15). Denote the masks of $\overline{{}_N\phi}_n(t)$ and $\widetilde{\widetilde{{}_{N,\widetilde{N}}\phi}}_n(t)$ by m_{N+n} and $\widetilde{m}_{N+n,\widetilde{N}-n}$, which can be found using Theorems 3.1 and 3.3. We have therefore given an easy way to derive sequences of pairs of biorthogonal spline type scaling functions shown in [2] and [4] (and in Section 2). For instance, the masks of ${}_1\phi$ and ${}_{1,3}\widetilde{\phi}$ are respectively

$${}_1m_0(\xi) = \frac{1+e^{-i\xi}}{2}$$

and

$${}_{1,3}\widetilde{m}_0(\xi) = -\frac{e^{2i\xi}}{16}(1+e^{-i\xi})^3(1-4e^{-i\xi}+e^{-2i\xi}).$$

Then the masks of $\overline{{}_1\phi}_n$ and $\widetilde{\widetilde{{}_{1,3}\phi}}_n$ for $n = 1, 2$ are respectively

$$\begin{aligned}
m_2(\xi) &= \frac{(1+e^{-i\xi})^2}{4},\ m_3(\xi) = \frac{(1+e^{-i\xi})^3}{8},\\
\widetilde{m}_{2,2}(\xi) &= -\frac{e^{i\xi}}{8}(1+e^{-i\xi})^2(1-4e^{-i\xi}+e^{-2i\xi}),\\
\widetilde{m}_{3,1}(\xi) &= -\frac{1}{4}(1+e^{-i\xi})(1-4e^{-i\xi}+e^{-2i\xi}),
\end{aligned}$$

which are respectively ${}_2m_0$, ${}_3m_0$, ${}_{2,2}\widetilde{m}_0$, and ${}_{3,1}\widetilde{m}_0$ (a different factor $e^{ik\xi}$ $(k \in \mathbb{Z})$ merely shifts the function support) shown in papers [3]

and [4]. In addition, by using Sweldens's lifting scheme (see [11]), we can construct masks ${}_{N,\widetilde{N}+2}\widetilde{m}_0$ from ${}_{N,\widetilde{N}}\widetilde{m}_0$. Therefore, starting from the mask of the Haar scaling function, ${}_1m_0 = {}_{1,1}\widetilde{m}_0$, we can obtain all ${}_nm_0$ $(n = 2, 3, \cdots)$ using Theorem 3.1 and all ${}_{n,\ell}\widetilde{m}_0$ using both the lifting scheme and the method supplied by Theorem 3.3. Specifically, from ${}_{1,1}\widetilde{m}_0$ we have all masks ${}_{1,2n+1}\widetilde{m}_0$ $(n = 1, 2, \cdots)$ by applying the lifting scheme, and all ${}_{\ell,2n+2-\ell}\widetilde{m}_0$ for $\ell = 2, 3, \cdots, 2n+1$ can be found by using the formulas in Theorem 3.3.

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Frame-based nonuniform sampling in Paley-Wiener spaces

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Abstract

This paper involves an extension of some ideas in papers of K. Gröchenig and his collaborators where frame-based algorithms are developed for the reconstruction of bandlimited signals on the line from nonuniformly spaced samples. We first develop a version of this algorithm which applies to certain important finite-dimensional subspaces of the Paley-Wiener space PW_Ω. Then it is shown how to extend these techniques to the approximate reconstruction of bandlimited, approximately timelimited signals from finitely many nonuniform samples. An estimate of the algorithm's complexity is also given.

1 Introduction

The classical space in which to study all aspects of sampling is the space V of finite energy signals "bandlimited" to $[-\Omega/2, \Omega/2]$:

$$V = \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \text{ for } |\xi| > \Omega/2\}.$$

Here the Fourier transform $\hat{f}$ of $f \in L^1(\mathbb{R})$ is given by $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2\pi it\xi}\, dt$ and if $\hat{f} \in L^1(\mathbb{R})$, then f may be recovered by $f(t) = (\hat{f})^\vee(t) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi it\xi}\, d\xi$. Because of its role in the celebrated Paley-Wiener theorem, V is often called the Paley-Wiener space and denoted PW_Ω. The Classical Sampling Theorem states that if $f \in PW_\Omega$, then f admits the expansion

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{\Omega}\right) \operatorname{sinc}(\Omega t - n) \tag{1}$$

where $\operatorname{sinc}(t) = \dfrac{\sin \pi t}{\pi t}$ is the *cardinal sine* function.

The Paley-Wiener spaces provide a good model for many phenomena. On the one hand, many natural and synthetic signal generators can only *output* slowly varying signals. On the other, many natural and synthetic processors can only *process* bandlimited signals. For example, the human ear cannot process/amplify signals beyond, say, 20 kHz. The Classical Sampling Theorem is fundamental in digital signal processing since it provides a means of converting an analog signal f to a digital signal $\{f(\frac{n}{\Omega})\}$ in a lossless way, and a means of reconstructing f continuously and uniquely within PW_Ω via (1).

The slow decay of the cardinal sine causes numerous problems in the application of (1). It is preferable in many respects to sample at a rate faster than

Ω samples per unit time (the *Nyquist rate*) if the cardinal sine function can be replaced in (1) by a function with more rapid decay. This can be done as follows ([2], [4], [15]).

Theorem 1. *Let $\varphi \in L^2(\mathbb{R})$ have Fourier transform $\hat{\varphi}$ satisfying*

$$\hat{\varphi}(\xi) = \begin{cases} 1 & \text{if } |\xi| \le \Omega/2 \\ \\ 0 & \text{if } |\xi| > \lambda\Omega/2 \end{cases} \tag{2}$$

for some $\lambda \ge 1$. Then each $f \in PW_\Omega$ admits the expansion

$$f(t) = \frac{1}{\lambda\Omega} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{\lambda\Omega}\right) \varphi\left(t - \frac{k}{\lambda\Omega}\right). \tag{3}$$

Proof. Since $\hat{f}$ is supported on $[-\lambda\Omega/2, \lambda\Omega/2]$, we have by the Fourier inversion theorem and the fact that $\{(\lambda\Omega)^{-1/2} e^{2\pi i k\xi/(\lambda\Omega)}\}_{k=-\infty}^{\infty}$ is an orthonormal basis for $L^2([-\lambda\Omega/2, \lambda\Omega/2])$,

$$\begin{aligned} \hat{f}(\xi) &= \sum_{k=-\infty}^{\infty} \frac{1}{\lambda\Omega} \left(\int_{-\lambda\Omega/2}^{\lambda\Omega/2} \hat{f}(\eta) e^{2\pi i k\eta/(\lambda\Omega)}\, d\eta \right) e^{-2\pi i k\xi/(\lambda\Omega)} \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{\lambda\Omega} f\left(\frac{k}{\lambda\Omega}\right) e^{-2\pi i k\xi/(\lambda\Omega)} \end{aligned}$$

for $|\xi| < \lambda\Omega$. Hence, since $\hat{\varphi}(\xi) = 1$ for $|\xi| \le \Omega/2$,

$$\hat{f}(\xi) = \hat{f}(\xi)\hat{\varphi}(\xi) = \sum_{k=-\infty}^{\infty} \frac{1}{\lambda\Omega} f\left(\frac{k}{\lambda\Omega}\right) e^{-2\pi i k\xi/(\lambda\Omega)} \hat{\varphi}(\xi)$$

and the result follows by taking the inverse Fourier transform of both sides of this last equation. □

It is a simple matter to construct functions satisfying (2). As an example, let B_N be the N-th order B-spline centered at 0, *i.e.*, if $\chi = \chi_{[-1/2,1/2]}$ then

$B_N = \chi * \chi * \cdots * \chi$ is the N-fold convolution product of χ with itself and $\widehat{B_N}(\xi) = \operatorname{sinc}^N(\xi)$. Further, B_N is supported on $[-N/2, N/2]$ and $\psi_N^\Omega(\xi) = \frac{2N}{\Omega} B_N\left(\frac{2N\xi}{\Omega}\right)$ is supported on $[-\Omega/4, \Omega/4]$. Finally,

$$\varphi_N^\Omega(t) = \frac{3\Omega}{2} \operatorname{sinc}\left(\frac{3\Omega t}{2}\right) \operatorname{sinc}^N\left(\frac{\Omega t}{2N}\right) \tag{4}$$

has Fourier transform $\widehat{\varphi_N^\Omega} = \chi_{[-3\Omega/4, 3\Omega/4]} * \psi_N^\Omega$ satisfying (2). Notice that $|\varphi_N^\Omega(t)| \le (2N/\Omega)^N (\pi|t|)^{-N-1}$. By choosing N large, the asymptotic decay of φ_N^Ω may be made arbitrarily fast in contrast with the $O(t^{-1})$ behavior of the cardinal sine as $t \to \infty$. However, for the algorithms we present in this paper, the improved rate of decay of φ_N^Ω is still insufficient. The growth of the constants in the decay estimates is also a concern in applications. In [21] it is shown how certain dilates of B-splines B_N converge to Gaussians along the lines of the Central Limit Theorem. Unfortunately, the dilation factors are inappropriate for our purposes.

There are fundamental constraints on the decay of functions whose Fourier transforms satisfy (2) that may be viewed as manifestations of the uncertainty principle. Firstly, if $\hat{\varphi}$ is to satisfy (2) then φ may not be compactly supported since by the Fourier inversion theorem it is clear that φ must be analytic on the entire complex plane. Also, φ cannot be dominated by a Gaussian, for if $|\varphi(t)| \le ce^{-at^2}$ for all t and any positive constants a and c, then the Fourier pair $(\varphi, \hat{\varphi})$ would violate Hardy's inequality [9], [12], [20]. If $|\varphi(t)| \le ce^{-a|t|}$ then by the Fourier inversion theorem, $\hat{\varphi}$ is analytic in the strip $|\operatorname{Im}(z)| < a$ and cannot therefore be compactly supported. Natterer [16] gives a function φ on the line with *root exponential decay* for which the Fourier transform $\hat{\varphi}$ satisfies (2):

Theorem 2. *Let ψ be given by its Fourier transform*

$$\hat{\psi}(\xi) = a \begin{cases} \exp\left(\frac{1}{\xi^2 - 1}\right) & \text{if } |\xi| < 1 \\ 0 & \text{else} \end{cases}$$

with $a = \left(\int_{-1}^{1} \exp\left(\frac{1}{\xi^2 - 1}\right) d\xi \right)^{-1}$. *Then*

$$|\psi(t)| \leq 16\sqrt{2}a\pi|t| \exp\left(1 + \frac{4}{\sqrt{2\pi|t|}} - \sqrt{2\pi|t|}\right) \quad \text{for } |t| > 1/2\pi.$$

Now let $\widehat{\psi_\Omega}(\xi) = \frac{4}{\Omega}\hat{\psi}\left(\frac{4\xi}{\Omega}\right)$ so that $\widehat{\psi_\Omega}$ is supported on $[-\Omega/4, \Omega/4]$. Define $\widehat{\varphi_\Omega}$ on the line by $\widehat{\varphi_\Omega} = \chi_{[-3\Omega/4, 3\Omega/4]} * \widehat{\psi_\Omega}$. Then $\widehat{\varphi_\Omega}$ satisfies (2) with $\lambda = 2$ and if $|t| > 1/(\pi\Omega)$,

$$|\varphi_\Omega(t)| = \frac{3\Omega}{2}|\text{sinc}\,(3\Omega t/2)||\psi_\Omega(t)| \leq ba\Omega \exp(-\sqrt{\pi\Omega|t|/2}) \tag{5}$$

with b a constant independent of Ω and a.

In [16], Natterer discusses the application of functions such as φ_Ω in the (uniform) oversampling result Theorem 1. While φ_Ω has improved decay over that enjoyed by the cardinal sine, or any of the spline examples (4), it suffers from a serious difficulty – namely that its exact values are unknown. Hence equation (3), although an attractive theoretical tool because of the rapid decay of φ_Ω, also poses computational difficulties. Natterer [16] employs piecewise rational approximations of φ_Ω to implement (3) while Strohmer [18] uses Gabor expansions with Gaussian windows.

As has already been observed, there is no (non-trivial) bandlimited function ψ on the line with $|\psi(t)| \leq ce^{-\alpha|t|}$ for any positive constant α. Natterer [16] conjectured the existence of non-trivial bandlimited functions $\psi_{\alpha,\sigma}$ with $\int \psi_{\alpha,\sigma} = 1$

and with *subexponential* decay, *i.e.*, which satisfy $|\psi_{\alpha,\sigma}(t)| \leq c_{\alpha,\sigma} e^{-\alpha|t|^\sigma}$ for all $\sigma < 1$. In an attempt to construct bandlimited wavelets with subexpontial decay, the conjecture was settled in the affirmative by Dziubański and Hernández [7] using results on the Gevrey class of functions. In fact, it is shown in [7] that given $\sigma < 1$, one may construct a function ψ which is bandlimited to $[-1/2, 1/2]$ such that $\psi(0) = 1$ and

$$|\psi(t)| \leq c 2^{1/(1-\sigma)} \exp(-|\pi t|^\sigma).$$

With $\psi_\Omega(t) = \psi\left(\frac{\Omega t}{2}\right)$ and $\varphi_\Omega(t) = \frac{3\Omega}{2}\mathrm{sinc}\left(\frac{3\Omega t}{2}\right)\psi_\Omega(t)$, we have

$$|\varphi_\Omega(t)| \leq \frac{c 2^{1/(1-\sigma)}}{|t|} \exp(-|\pi\Omega t|^\sigma), \tag{6}$$

and φ_Ω satisfies (2) with $\lambda = 2$.

While Theorem 1 and the work of Natterer and Strohmer effectively solve the computational issues that arise due to the slow decay of the cardinal sine, in many applications sampled data is not given uniformly. It is important then that effective algorithms be developed for the reconstruction of bandlimited signals from nonuniformly sampled data. Of course, to be computationally feasible, such algorithms can handle only a finite amount of data. However, the finite data requirement cannot be achieved with perfect reconstruction.

Frames have long been recognized as useful in dealing with data obtained from nonuniform sampling of bandlimited signals. Seminal results on the relationship between the density of a sampling set $\Lambda \subset \mathbb{R}$ and the frame bounds associated with collections of translated cardinal sines $\{\mathrm{sinc}\,\Omega(t-\lambda)\}_{\lambda\in\Lambda}$ are reviewed in this paper – see, for example, the result of Duffin and Schaeffer

(Theorem 3) and that of Seip (Theorem 4). Gröchenig and his collaborators [1], [11], [18], [19] constructed frames for PW_Ω which are adapted to the nonuniform local geometry of the sampling set. The algorithm reconstructs $f \in PW_\Omega$ from nonuniform samples $\{f(\lambda)\}_{\lambda\in\Lambda}$ in an iterative manner. Since PW_Ω is infinite-dimensional, no finite collection of samples $\{f(\lambda_j)\}_{j=1}^N$ is sufficient to determine f, let alone be sufficient for reconstruction. Consequently, each iteration of the algorithm requires an infinite number of samples be taken, and an infinite sum be performed – requirements which are computationally unfeasible. Other methods ([11], [18]) use approximations of signals in PW_Ω by trigonometric polynomials of appropriate degree followed by the application of frame algorithms for the reconstruction of trigonometric polynomials from a finite number of samples. Of course, the reconstruction approximates the original signal no more closely than does the approximating trigonometric polynomial. Furthermore, the trigonometric polynomial approximation is periodic so that the reconstruction will be poor away from the region in which the samples were taken.

This paper represents an attempt to adapt the Gröchenig algorithm to handle finite samples. Rather than approximate $f \in PW_\Omega$ by a trigonometric polynomial, it is proposed that the approximation be performed in a certain natural, finite-dimensional subspace PW_Ω^M of PW_Ω. Finite frame algorithms for PW_Ω^M are then applied to obtain an approximate reconstruction of f from finite samples. The use of PW_Ω^M rather than trigonometric polynomials appears to be more natural, and gives better behavior of the reconstruction away from

the sampling points.

This paper is organized as follows. In Section 2 we outline the very basic techniques of frame theory needed to understand the algorithms of later sections. Section 3 contains a discussion of results in nonuniform sampling of bandlimited signals, from the perspective of frame theory. In particular, separation and density conditions on sampling sets which ensure the convergence of frame-based reconstruction algorithms are briefly surveyed. An *adaptive* frame algorithm (Theorem 5) due to Aldroubi and Gröchenig is presented. Such algorithms give perfect reconstructions and therefore require an infinite amount of data. Section 4 introduces the finite-dimensional model subspace PW_Ω^M of PW_Ω in which we work. Theorem 6 is a variation on Theorem 5 in which it is shown that $f \in PW_\Omega^M$ can be reconstructed perfectly (though iteratively) via a frame algorithm, with finite data. The complexity of the algorithm is estimated. Finally, in Section 5 the space $\mathcal{F}(\Omega, t; \varepsilon)$ of bandlimited ε-timelimited signals is introduced. Through an uncertainty inequality due to Nazarov (Theorem 7) an appropriate relationship between $\mathcal{F}(\Omega, T; \varepsilon)$ and PW_Ω^M is obtained (Lemma 1). This leads, through several intermediate results, to an algorithm for the *approximate* reconstruction of signals $f \in PW_\Omega$ from finitely many nonuniform samples. An estimate of the complexity of the algorithm is given, incorporating the bandlimit Ω, an approximate timelimit T of f, and the approximation error ε.

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2 Frames and the frame algorithm

The aim of this section is to lay down those fundamental aspects of frame theory that are of direct relevance to this paper. For more details, the reader is referred to the excellent books of Christensen [5] and Daubechies [6] and the tutorial article [14] by Heil and Walnut.

Let $\mathcal{H}$ be a (separable) Hilbert space with inner product $\langle \cdot, \cdot \rangle$. A sequence $\{e_n\}_{n=0}^{\infty} \subset \mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{n=0}^{\infty} |\langle f, e_n \rangle|^2 \leq B\|f\|^2 \tag{7}$$

for all $f \in \mathcal{H}$. The constants A and B are the lower and upper *frame bounds* respectively. Associated with each frame is the *analysis (digitization) operator* $D : \mathcal{H} \to \ell^2(\mathbb{Z}_+)$ given by $Df = \{\langle f, e_n \rangle\}_{n=0}^{\infty}$ and its Hilbert space adjoint $S = D^* : \ell^2(\mathbb{Z}_+) \to \mathcal{H}$ which is known as the *synthesis operator* and is given by $Sc = \sum_{n=0}^{\infty} c_n e_n$. The *frame operator* is the self-adjoint composition $T = S \circ D : \mathcal{H} \to \mathcal{H}$ given by $Tf = \sum_{n=0}^{\infty} \langle f, e_n \rangle e_n$. Equation (7) may be written as

$$A\|f\|^2 \leq \langle Tf, f \rangle \leq B\|f\|^2 \tag{8}$$

i.e., $AI \leq T \leq BI$, so that T is invertible with $B^{-1}I \leq T^{-1} \leq A^{-1}I$. Each

$f \in \mathcal{H}$ admits the *frame expansion*

$$f = TT^{-1}f = \sum_{n=0}^{\infty} \langle T^{-1}f, e_n \rangle e_n = \sum_{n=0}^{\infty} \langle f, g_n \rangle e_n \tag{9}$$

with $g_n = T^{-1}e_n$. The collection $\{g_n\}_{n=0}^{\infty}$ is itself a frame (the *dual frame*). Similarly we have

$$f = T^{-1}Tf = \sum_{n=0}^{\infty} \langle f, e_n \rangle g_n. \tag{10}$$

2.1 The frame algorithm

Computation of the dual elements g_n in (9) or (10) may be difficult, and desirable properties of the original frame elements such as smoothness and rapid decay may not be inherited by elements of the dual frame. Elementary results from functional analysis can be used to generate iterative algorithms for reconstruction of $f \in \mathcal{H}$ from its frame coefficients $\langle f, e_n \rangle$ without reference to the dual frame.

By (8), $|\langle (I - 2T/(A+B))f, f \rangle| \leq ((B-A)/(B+A))\|f\|^2$ so that by the self-adjointness of T,

$$\left\| I - \frac{2}{A+B}T \right\| < \left(\frac{B-A}{B+A} \right) < 1. \tag{11}$$

Hence, the inverse of T may be computed via the convergent *Neumann series*

$$T^{-1} = \frac{2}{A+B} \sum_{j=0}^{\infty} \left(I - \frac{2}{A+B}T \right)^j. \tag{12}$$

Since $f = T^{-1}Tf$, (12) gives

$$f = T^{-1}Tf = \frac{2}{A+B} \sum_{j=0}^{\infty} \left(I - \frac{2}{A+B}T \right)^j Tf \tag{13}$$

cf. (10). Letting f_n be the n-th partial sum on the right-hand side of (13) gives the iteration

$$f_0 = \frac{2}{A+B}Tf; \quad f_{n+1} = f_n + \frac{2}{A+B}T(f - f_n). \tag{14}$$

Notice that by (11) and (8), the error $f - f_n$ satisfies

$$\begin{aligned} \|f - f_n\| &= \left\| \frac{2}{A+B} \sum_{j=n+1}^{\infty} \left(I - \frac{2}{A+B}T \right)^j Tf \right\| \\ &\leq \frac{2}{A+B} \sum_{j=n+1}^{\infty} \left(\frac{B-A}{B+A} \right)^j B\|f\| = \kappa \left(\frac{\kappa - 1}{\kappa + 1} \right)^{n+1} \|f\| \end{aligned} \tag{15}$$

where $\kappa = B/A$ is the *condition number* of the frame, so that $f_n \to f$ geometrically as $n \to \infty$ with the rate of convergence determined by κ.

There are two difficulties that arise in the application of (14) to the reconstruction of $f \in \mathcal{H}$ from data $\{\langle f, e_n \rangle\}_{n=0}^{\infty}$. Notice that if the frame is *tight* (*i.e.*, if $\kappa = 1$) then by (15), $f = f_0 = (2/(A+B)) \sum_{n=0}^{\infty} \langle f, e_n \rangle e_n$, a formula reminiscent of an expansion in an orthonormal basis. However, if κ is large, so that $\rho = (\kappa - 1)/(\kappa + 1) \approx 1$, then the convergence $f_n \to f$ is slow. In fact, the number of iterations required to ensure that $\|f - f_n\| \leq \varepsilon \|f\|$ is $n \geq (\log(1/\varepsilon) + \log \kappa)/\log(1/\rho)$ which is of course large for $\rho \approx 1$.

The second difficulty is that applying (14) assumes knowledge of the frame bounds A and B. In fact there are many examples of systems for which the existence of frame bounds is known, while their precise value is not. See, for example, Theorem 4. In these situations, the reconstruction (14) is useless. Poorly conditioned frames (those for which $\kappa = B/A \gg 1$) provide slow convergence of the iteration (14). Relaxation and acceleration techniques [10], [15]

help alleviate these problems but their consideration is beyond the scope of this paper.

3 Nonuniform sampling in PW_Ω

Sets $\{\lambda_n\}_{n=-\infty}^{\infty} \subset \mathbb{R}$ for which the collection $e_n(\xi) = e^{2\pi i \lambda_n \xi}$ $(n \in \mathbb{Z},\ |\xi| \le \Omega/2)$ is a frame for $L^2([-\Omega/2, \Omega/2])$ are crucial in the sampling theory of Paley-Wiener spaces. In fact, if $\varphi_n(t) = \Omega \operatorname{sinc} \Omega(t - \lambda_n)$ $(n \in \mathbb{Z})$ and $f \in PW_\Omega$, then by the Parseval relation,

$$\langle f, \varphi_n \rangle = \int_{-\Omega/2}^{\Omega/2} \hat{f}(\xi) \overline{e_n}(\xi)\, d\xi = f(\lambda_n)$$

and therefore

$$\sum_{n=-\infty}^{\infty} |f(\lambda_n)|^2 = \sum_{n=-\infty}^{\infty} |\langle f, \varphi_n \rangle|^2 = \sum_{n=-\infty}^{\infty} |\langle \hat{f}, e_n \chi_{[-\Omega/2, \Omega/2]} \rangle|^2$$

so that $\{e_n\}_{n=-\infty}^{\infty}$ is a frame for $L^2([-\Omega/2, \Omega/2])$ with frame bounds A and B if and only if $\{\varphi_n\}_{n=-\infty}^{\infty}$ is a frame for PW_Ω with the same frame bounds, which in turn is equivalent to

$$A\|f\|_2^2 \le \sum_{n=-\infty}^{\infty} |f(\lambda_n)|^2 \le B\|f\|_2^2$$

for all $f \in PW_\Omega$. In this case, signals $f \in PW_\Omega$ may be reconstructed from samples $f(\lambda_n)$ using standard frame algorithms.

The frame radius $R = R(\Lambda)$ of a subset $\Lambda \subset \mathbb{R}$ is defined to be

$$R(\Lambda) = \sup\{r :\ \{e^{2\pi i \lambda \xi}\}_{\lambda \in \Lambda} \text{ is a frame for } L^2([-r, r])\}.$$

A subset $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty} \subset \mathbb{R}$ is *(uniformly) separated* if $\inf_{j\neq k} |\lambda_j - \lambda_k| = \alpha > 0$ in which case α is said to be the *s*eparation constant of Λ. If Λ is separated, it has *uniform density* d if there exists $L > 0$ such that $|\lambda_k - k/d| \leq L$ for all k.

An seminal result on frames of exponentials is due to Duffin and Schaeffer [8]:

Theorem 3. *If Λ is separated with uniform density $d > 0$ then $R(\Lambda) \geq d/2$.*

In particular, if $d \geq \Omega$, then $\{\operatorname{sinc}\Omega(\cdot - \lambda_k)\}_{k=-\infty}^{\infty}$ is a frame for PW_Ω. A more refined result uses the lower Beurling densities and is outlined below in Theorem 4.

Given a discrete set $\Lambda \subset \mathbb{R}$, let

$$n_\Lambda^+(r) = \sup_{x\in\mathbb{R}} |\{\Lambda \cap [x - r/2, x + r/2]\}|; \quad n_\Lambda^-(r) = \inf_{x\in\mathbb{R}} |\{\Lambda \cap [x - r/2, x + r/2]\}|,$$

where $|A|$ denotes the cardinality of the discrete set A. Then the upper and lower Beurling densities of Λ are, respectively,

$$\mathcal{D}^+(\Lambda) = \lim_{r\to\infty} \frac{n_\Lambda^+(r)}{r}; \quad \mathcal{D}_\Lambda^-(r) = \lim_{r\to\infty} \frac{n_\Lambda^-(r)}{r}.$$

A sequence Λ is *relatively separated* if it is the finite union of separated sequences.

The following result is due to Seip [17].

Theorem 4. *If $\{e^{2\pi i\lambda_n\xi}\}_{\lambda_n\in\Lambda}$ is a frame for $L^2([-\Omega/2, \Omega/2])$, then Λ is relatively separated and $\mathcal{D}^-(\Lambda) \geq \Omega$. On the other hand, if Λ is relatively separated and $\mathcal{D}^-(\Lambda) > \Omega$, then $\{e^{2\pi i\lambda_n\xi}\}_{\lambda_n\in\Lambda}$ is a frame for $L^2([-\Omega/2, \Omega/2])$.*

Suppose $\{e^{2\pi i\lambda_n\xi}\}_{\lambda_n\in\Lambda}$ is a frame for $L^2([-\Omega/2, \Omega/2])$. Then the collection $\{\varphi_n(t) = \Omega \operatorname{sinc}\Omega(t - \lambda_n)\}_{\lambda_n\in\Lambda}$ is a frame for PW_Ω and we may apply (10) to

reconstruct signals in PW_Ω. In fact, since $\langle f, \varphi_n \rangle = f(\lambda_n)$ for $f \in PW_\Omega$, we have

$$f(t) = \sum_n f(\lambda_n) g_n(t)$$

with $\{g_n\}_n$ the dual frame to $\{\varphi_n\}_n$. By (12), each dual frame element may be computed by

$$g_n = \frac{2}{A+B} \sum_{j=0}^{\infty} \left(I - \frac{2T}{A+B} \right)^j \varphi_n.$$

To make the calculation computationally feasible, this sum must be truncated, but it is important that the truncation be performed in such a way as to ensure high accuracy. Hence we compute

$$\tilde{g}_n = \frac{2}{A+B} \sum_{j=0}^{N(n)} \left(I - \frac{2T}{A+B} \right)^j \varphi_n$$

for $N(n)$ sufficiently large. Then an approximation of f is given by

$$\tilde{f} = \sum_{n=-M}^{M} f(\lambda_n) \tilde{g}_n \tag{16}$$

for M sufficiently large so that $\|f - \tilde{f}\|_2^2 < \varepsilon \|f\|_2^2$ for a prescribed accuracy ε. Unfortunately this method proves to be computationally expensive and is further complicated by the absence of any information regarding the frame bounds. It is therefore preferable in many respects to use the frame algorithm for reconstruction since it avoids construction of the dual frame, although this too is problematic when the frame bounds are unknown. For this reason, as well as for accelerated convergence, the frame conjugate gradient algorithm is preferred [10], [15].

Gröchenig and his collaborators (see [1] and references therein) proposed the use of *adaptive frames* for the reconstruction of bandlimited signals from

nonuniformly sampled data. The adaptivity refers to the way the frame elements are adapted to the local geometry of the sampling set Λ. We state a simplified version of a result in [1]:

Theorem 5. *Suppose* $\Lambda = \{\lambda_l\}_{l=-\infty}^{\infty} \subset \mathbb{R}$ *satisfies:*

(i) $\lambda_l < \lambda_{l+1}$ *for all* l, $\lambda_l \to \pm\infty$ *as* $l \to \pm\infty$, *and*

(ii) $\delta = \sup_l |\lambda_{l+1} - \lambda_l| < 1/\Omega$.

Let $v_l = (\lambda_{l+1} - \lambda_{l-1})/2$ *and* $\varphi_l(t) = \sqrt{v_l}\, sinc\, \Omega(t - \lambda_l)$ $(l \in \mathbb{Z})$. *Then* $\{\varphi_l\}_{l=-\infty}^{\infty}$ *is a frame for* PW_Ω *with lower frame bound* $A = (1 - \delta\Omega)^2$ *and upper frame bound* $B = (1 + \delta\Omega)^2$.

Note that in Theorem 5, the sampling set Λ need not be relatively separated (cf. Theorem 4). The factor $\sqrt{v_l}$ in the definition of the frame elements φ_l is the source of the aforementioned adaptivity of this scheme. Observe also that v_l is the length of the Voronoi region surrounding the lattice point λ_l.

In the context of the theorem, the frame iteration (14) becomes

$$f_0(t) = \frac{1}{(1 + \delta^2\Omega^2)} \sum_n v_n f(\lambda_n) \operatorname{sinc} \Omega(t - \lambda_n), \tag{17}$$

$$f_{j+1}(t) = f_j(t) + \frac{1}{(1 + \delta^2\Omega^2)} \sum_n v_n (f(\lambda_n) - f_j(\lambda_n)) \operatorname{sinc} \Omega(t - \lambda_n), \tag{18}$$

and converges in PW_Ω to f with $\|f - f_j\|_2 \le (\delta\Omega)^{j+1}(1 + \delta\Omega)\|f\|_2$ by (15).

By the Classical Sampling Theorem, complete knowledge of $f \in PW_\Omega$ may be obtained from the uniform samples $f(k/\Omega)$ $(k \in \mathbb{Z})$. If $a_k^{(j)} = f_j(k/\Omega)$, then

from (17) and (18) we have

$$\begin{aligned} a_k^{(0)} &= \frac{1}{1+\delta^2\Omega^2}\sum_n v_n f(\lambda_n)\,\mathrm{sinc}\,(\Omega\lambda_n - k), \\ a_k^{(j+1)} &= a_k^{(j)} + a_k^{(0)} - \sum_l F_{kl} a_l^{(j)}, \end{aligned}$$

where $F_{kl} = \dfrac{1}{1+\delta^2\Omega^2}\sum_n v_n\,\mathrm{sinc}\,(\Omega\lambda_n - l)\,\mathrm{sinc}\,(\Omega\lambda_n - k)$. While these iterations solve the problems associated with the nonuniformity of the data, again the slow decay of the cardinal sine function means that these infinite sums cannot be accurately truncated with reasonable computational cost for prescribed accuracy.

4 Nonuniform sampling in PW_Ω^M

In this section we develop a frame-based sampling technique for signals in an appropriate finite-dimensional subspace of PW_Ω.

Definition 1. *With $\Omega > 0$ and M a positive integer, define PW_Ω^M to be the subspace of PW_Ω given by*

$$PW_\Omega^M = span\,\{sinc\,(\Omega t - k)\}_{k=-M}^{M}.$$

Equivalently, $f \in PW_\Omega^M \iff f \in L^2(\mathbb{R})$ and

$$\hat{f}(\xi) = \chi_{[-\Omega/2,\Omega/2]}(\xi)\sum_{j=-M}^{M} a_j e^{2\pi i j\xi/\Omega}$$

for some $\{a_j\}_{j=-M}^{M} \subset \mathbb{C}$. Alternatively, PW_Ω^M may be thought of as the class of those signals $f \in PW_\Omega$ for which $f(k/\Omega) = 0$ for all $|k| > M$.

The dimension of PW_Ω^M is $2M+1$ since $\{\operatorname{sinc}\Omega(t-k/\Omega)\}_{k=-M}^{M}$ is an orthonormal basis for this space.

As in [1], a key ingredient in the sampling results we present is Wirtinger's inequality: if $f \in L^2(\mathbb{R})$, $a, b \in \mathbb{R}$ and $f(a)f(b) = 0$, then

$$\int_a^b |f(t)|^2\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b |f'(t)|^2\,dt. \tag{19}$$

For any sampling scheme to be truly implementable, it must use a finite number of samples $\{f(\lambda_n)\}_{n=-N}^{N}$. In what follows we insist that the sampling points $\Lambda_N = \{\lambda_n\}_{n=-N}^{N}$ be drawn from an infinite set of sampling points $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty}$ satisfying the conditions of Theorem 5.

Theorem 6. *Fix a positive integer M, $\Omega > 0$ and let $\Lambda = \{\lambda_l\}_{l=-\infty}^{\infty} \subset \mathbb{R}$ be a lattice satisfying conditions (i) and (ii) of Theorem 5 as well as*

(iii) *$n_\Lambda^+(r) < \alpha r$ for some $r \ge 2M/\Omega$, $\alpha \ge 1/\delta$, and*

(iv) *$|\lambda_0| \le |\lambda_l|$ for all l.*

Let $v_l = (\lambda_{l+1} - \lambda_{l-1})/2$ and P_Ω^M be the orthogonal projection of PW_Ω onto PW_Ω^M. Then for all $\sigma \in [1/2, 1)$ and $\omega \in (0, 1-\delta^2\Omega^2)$, there exists a constant C independent of $\sigma, M, \Omega, \delta, \alpha, \omega$ such that if

$$N \ge \frac{(2\alpha\delta - 1)}{\delta\Omega}\left(M + \frac{1}{2} + \frac{8(M+1)}{\pi^2\omega} + \frac{2^{2/\sigma}}{\pi}\left[\left(\frac{C}{\omega}\right)^{1/\sigma} + \left(\frac{2}{1-\sigma}\right)^{1/\sigma}\right]\right) + \frac{1}{\delta},$$

then the collection

$$\varphi_l^M(t) = \sqrt{v_l}P_\Omega^M(sinc\,\Omega(t-\lambda_l)) \qquad (|l| \le N)$$

is a frame for PW_Ω^M.

Note that the frame has frame bounds which we estimate in the proof.

Proof. Define η_j $(-N \le j \le N+1)$ by $\eta_{-N} = \lambda_{-N}$, $\eta_{N+1} = \lambda_N$ and $\eta_j = (\lambda_j + \lambda_{j-1})/2$ $(1-N \le j \le N)$. Define the *sample and hold operator* Q_{Λ_N} on $C(\mathbb{R})$ by $Q_{\Lambda_N} f(t) = \sum_{l=-N}^{N} f(\lambda_l)\chi_{[\eta_l,\eta_{l+1})}(t)$. Then, with $\beta_N = \min\{\lambda_N, -\lambda_{-N}\}$,

$$\begin{aligned}\|f - Q_{\Lambda_N} f\|_2^2 &= \int_{\lambda_{-N}}^{\lambda_N} |f(t) - Q_{\Lambda_N} f(t)|^2\,dt + \int_{\mathbb{R}\setminus[\lambda_{-N},\lambda_N]} |f(t)|^2\,dt \\ &\le \int_{\lambda_{-N}}^{\lambda_N} |f(t) - Q_{\Lambda_N} f(t)|^2\,dt + \int_{|t|>\beta_N} |f(t)|^2\,dt = I + II. \qquad (20)\end{aligned}$$

From now on we assume $f \in PW_\Omega$. Integral I in (20) may be estimated as in [1] with the help of Wirtinger's inequality (19):

$$\begin{aligned} I &= \sum_{l=-N}^{N} \int_{\eta_l}^{\eta_{l+1}} |f(t) - f(\lambda_l)|^2\,dt \\ &= \sum_{l=-N}^{N} \left(\int_{\eta_l}^{\lambda_l} |f(t) - f(\lambda_l)|^2\,dt + \int_{\lambda_l}^{\eta_{l+1}} |f(t) - f(\lambda_l)|^2\,dt \right) \\ &\le \left(\frac{\delta}{\pi}\right)^2 \int_{\lambda_{-N}}^{\lambda_N} |f'(t)|^2\,dt \\ &\le \left(\frac{\delta}{\pi}\right)^2 \int_{-\Omega/2}^{\Omega/2} 4\pi^2\xi^2 |\hat{f}(\xi)|^2\,d\xi \le \delta^2\Omega^2\|f\|_2^2. \qquad (21)\end{aligned}$$

Integral II requires a different technique. We employ the oversampling representation (3) with $\lambda = 2$ and $\varphi = \varphi_\Omega$ having subexponential decay of the form (6). The point is that σ may be chosen to be arbitrarily close to 1. For a positive integer L to be determined later:

$$\begin{aligned} II &\le \frac{1}{2\Omega^2} \int_{|t|>\beta_N} \left| \sum_{|k|\le L} f\left(\frac{k}{2\Omega}\right) \varphi_\Omega\left(t - \frac{k}{2\Omega}\right) \right|^2 dt \\ &+ \frac{1}{2\Omega^2} \int_{|t|>\beta_N} \left| \sum_{|k|> L} f\left(\frac{k}{2\Omega}\right) \varphi_\Omega\left(t - \frac{k}{2\Omega}\right) \right|^2 dt = J_1 + J_2. \end{aligned}$$

An application of Cauchy-Schwarz to the sum in integral J_1 and the subexponential decay of φ_Ω gives

$$\begin{aligned} J_1 &\leq \frac{1}{2\Omega^2}\sum_{|k|\leq L}\left|f\Big(\frac{k}{2\Omega}\Big)\right|^2 \sum_{|l|\leq L}\int_{|t|>\beta_N}\left|\varphi_\Omega\Big(t-\frac{l}{2\Omega}\Big)\right|^2 dt \\ &\leq C2^{2/(1-\sigma)}\Omega\|f\|_2^2\sum_{|l|\leq L}\int_{|t|>\beta_N}\exp\Big(-2\Big|\pi\Omega\Big(t-\frac{l}{2\Omega}\Big)\Big|^\sigma\Big)\,dt \\ &\leq C2^{2/(1-\sigma)}\|f\|_2^2\sum_{|l|\leq L}\exp(-|\pi(2\Omega\beta_N-l)/2|^\sigma) \\ &\leq C2^{2/(1-\sigma)}\|f\|_2^2\int_{-L}^{L+1}\exp(-|\pi(2\Omega\beta_N-t)/2|^\sigma)\,dt \\ &\leq C2^{2/(1-\sigma)}\|f\|_2^2\exp\Big(-\frac{1}{2}|\pi(2\Omega\beta_N-L-1)/2|^\sigma\Big). \end{aligned} \tag{22}$$

It is important to note here that the constant C in (22) is independent of σ since it is of the form $C=Bc^2$ with B independent of sigma (since $\sigma\geq 1/2$) and c the σ-independent constant of (6). On the other hand, with $g_k = f\Big(\frac{k}{2\Omega}\Big)\chi_{|k|>L}$, the Plancherel theorem gives

$$\begin{aligned} J_2 &= \frac{1}{2\Omega^2}\int_{|t|>\beta_N}\left|\sum_{k=-\infty}^{\infty} g_k\varphi_\Omega\Big(t-\frac{k}{2\Omega}\Big)\right|^2 dt \\ &\leq \frac{1}{2\Omega^2}\int_{-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty} g_k\varphi_\Omega\Big(t-\frac{k}{2\Omega}\Big)\right|^2 dt \\ &= \frac{1}{2\Omega^2}\int_{-\infty}^{\infty}\left|G\Big(\frac{\xi}{2\Omega}\Big)\right|^2|\hat{\varphi}_\Omega(\xi)|^2\,d\xi \end{aligned}$$

where $G(\xi) = \sum_{k=-\infty}^{\infty} g_k e^{-2\pi i k\xi}$ is periodic on the line with period 1. Therefore,

$$\begin{aligned} J_2 &\leq \frac{1}{2\Omega^2} \sum_{l=-\infty}^{\infty} \int_{2\Omega l}^{2\Omega(l+1)} \left| G\left(\frac{\xi}{2\Omega}\right) \right|^2 |\hat{\varphi}_\Omega(\xi)|^2 \, d\xi \\ &= \frac{1}{2\Omega^2} \int_0^{2\Omega} \left| G\left(\frac{\xi}{2\Omega}\right) \right|^2 \sum_{l=-\infty}^{\infty} |\hat{\varphi}_\Omega(\xi + 2\Omega l)|^2 \, d\xi \\ &\leq \frac{1}{2\Omega^2} \sup_\eta \sum_{l=-\infty}^{\infty} |\hat{\varphi}_\Omega(\eta + 2\Omega l)|^2 \int_0^{2\Omega} \left| G\left(\frac{\xi}{2\Omega}\right) \right|^2 d\xi \\ &= \frac{1}{\Omega} \|g\|_{\ell^2}^2 \sup_\eta \sum_{l=-\infty}^{\infty} |\hat{\varphi}_\Omega(\eta + 2\Omega l)|^2. \end{aligned}$$

Now $0 \leq \hat{\varphi}_\Omega(\xi) \leq 1$ for all ξ and $\hat{\varphi}_\Omega(\xi) = 0$ for $|\xi| > \Omega$, so $\sum_{l=-\infty}^{\infty} |\hat{\varphi}_\Omega(\eta + 2\Omega l)|^2 \leq 1$ for all η, and

$$J_2 \leq \frac{1}{\Omega} \|g\|_{\ell^2}^2 = \frac{1}{\Omega} \sum_{|k|>L} \left| f\left(\frac{k}{2\Omega}\right) \right|^2. \tag{23}$$

However, we may apply the Classical Sampling Theorem to obtain

$$f\left(\frac{k}{2\Omega}\right) = \sum_{j=-M}^{M} f\left(\frac{j}{\Omega}\right) \operatorname{sinc}\left(j - \frac{k}{2}\right),$$

and substituting this into (23) gives

$$\begin{aligned} J_2 &\leq \frac{1}{\Omega} \sum_{j=-M}^{M} \left| f\left(\frac{j}{\Omega}\right) \right|^2 \sum_{l=-M}^{M} \sum_{|k|>L} \operatorname{sinc}^2\left(j - \frac{k}{2}\right) \\ &\leq \frac{4\|f\|_2^2}{\pi^2} \sum_{j=-M}^{M} \sum_{|k|>L} \frac{1}{(2j-k)^2} \\ &= \frac{8\|f\|_2^2}{\pi^2} \sum_{j=-M}^{M} \frac{L}{L^2/4 - j^2} \\ &\leq \frac{2\|f\|_2^2}{\pi^2} \left(\frac{4}{L} + \int_{-M}^{M} \frac{L}{L^2/4 - x^2} \, dx \right) \\ &= \frac{2\|f\|_2^2}{\pi^2} \left(\frac{4}{L} + \log\left(\frac{L+2M}{L-2M}\right) \right). \end{aligned} \tag{24}$$

Combining (20), (21), (22) and (24) gives

$$\|f - Q_{\Lambda_N} f\|_2 \leq \gamma \|f\|_2$$

with

$$\gamma^2 = \delta^2\Omega^2 + \rho + \theta \tag{25}$$

where

$$\rho = C2^{2/(1-\sigma)} \exp\left(\frac{-1}{2}|\pi(2\Omega\beta_N - L - 1)/2|^\sigma\right), \quad \theta = \frac{2}{\pi^2}\left(\frac{4}{L} + \log\left(\frac{L+2M}{L-2M}\right)\right).$$

The collection $\{\varphi_j^M\}_{j=-N}^N$ is a frame provided $\gamma < 1$. Therefore we require $\rho + \theta < \omega$ for $0 < \omega < 1 - \delta^2\Omega^2$. This can be achieved, for example, by insisting ρ, $\theta < \omega/2$. However,

$$\theta < \frac{\omega}{2} \iff \frac{4}{L} + \log\left(1 + \frac{4M}{L-2M}\right) < \frac{\pi^2\omega}{4}.$$

Since $\log(1+x) < x$ for all positive x, it is sufficient to insist that $\frac{1}{L} + \frac{M}{L-2M} < \frac{\pi^2\omega}{16}$ which in turn may be achieved by insisting that

$$L > 2M + \frac{16(M+1)}{\pi^2\omega}. \tag{26}$$

On the other hand,

$$\rho < \frac{\omega}{2} \iff \Omega\beta_N > \frac{L+1}{2} + \frac{2^{1/\sigma}}{\pi}\log^{1/\sigma}\left(\frac{2^{(3-\sigma)/(1-\sigma)}C}{\omega}\right)$$

which, by (26) requires

$$\Omega\beta_N > M + \frac{1}{2} + \frac{8(M+1)}{\pi^2\omega} + \frac{2^{1/\sigma}}{\pi}\log^{1/\sigma}\left(\frac{2^{(3-\sigma)/(1-\sigma)}C}{\omega}\right). \tag{27}$$

Observe that since $n_\Lambda^+(r) < \alpha r$ for $r > 2M/\Omega$, the lattice has upper Beurling density $\mathcal{D}_\Lambda^+ \leq \alpha$. However, the density condition $|\lambda_{l+1} - \lambda_l| \leq \delta$ implies that $\mathcal{D}_\Lambda^+ \geq \delta^{-1}$, hence that $\alpha\delta \geq 1$. Since by (27) $2\beta_N > 2M/\Omega$, we have $n_\Lambda(2\beta_N) < 2\alpha\beta_N$, *i.e.*,

$$|\{\Lambda \cap [-\beta_N, \beta_N]\}| < 2\alpha\beta_N. \tag{28}$$

Without loss of generality, $\lambda_N \leq -\lambda_{-N}$ so that $\beta_N = \lambda_N$. Then, since $|\lambda_{l+1} - \lambda_l| \leq \delta$, we have

$$|\{\Lambda \cap [-\beta_N, \beta_N]\}| \geq N + \frac{\beta_N - 1}{\delta}. \tag{29}$$

Combining (28) and (29) gives

$$\Omega\beta_N \geq \left(N - \frac{1}{\delta}\right)\left(\frac{\Omega\delta}{2\alpha\delta - 1}\right).$$

To achieve (27) it is therefore sufficient that

$$\left(N - \frac{1}{\delta}\right)\left(\frac{\Omega\delta}{2\alpha\delta - 1}\right) \geq M + \frac{1}{2} + \frac{8(M+1)}{\pi^2\omega} + \frac{2^{1/\sigma}}{\pi}\log^{1/\sigma}\left(\frac{2^{(3-\sigma)/(1-\sigma)}C}{\omega}\right).$$

Since $\log(x) \leq x$, it is sufficient to have N bounded below by

$$N \geq \frac{(2\alpha\delta - 1)}{\delta\Omega}\left(M + \frac{1}{2} + \frac{8(M+1)}{\pi^2\omega} + \frac{2^{2/\sigma}}{\pi}\left[\left(\frac{C}{\omega}\right)^{1/\sigma} + \left(\frac{2}{1-\sigma}\right)^{1/\sigma}\right]\right) + \frac{1}{\delta}. \tag{30}$$

Hence, if N is bounded below as in (30), then

$$(1 - \sqrt{\delta^2\Omega^2 + \omega})^2\|f\|_2^2 \leq \|Q_{\Lambda_N} f\|_2^2 \leq (1 + \sqrt{\delta^2\Omega^2 + \omega})^2\|f\|_2^2. \tag{31}$$

Notice that for all $f \in PW_\Omega^M$, $\sum_{j=-N}^{N} |\langle f, \varphi_j^M\rangle|^2 = \|Q_{\Lambda_N} f\|_2^2$, so that by (31), $\{\varphi_j^M\}_{j=-N}^N$ is a frame for PW_Ω^M with upper and lower frame bounds $B = (1 + \sqrt{\delta^2\Omega^2 + \omega})^2$ and $A = (1 - \sqrt{\delta^2\Omega^2 + \omega})^2$ respectively. In fact, the upper frame bound B may be taken as $B = (1 + \delta\Omega)^2$. □

4.1 Frame algorithm

Consider the frame $\{\varphi_l^M\}_{l=-N}^N$ of Theorem 6. The frame operator T_Ω^M takes the form

$$T_\Omega^M f(t) = \sum_{j=-M}^{M}\left(\sum_{l=-N}^{N} v_l f(\lambda_l)\,\mathrm{sinc}\,(j - \Omega\lambda_l)\right)\mathrm{sinc}\,(\Omega t - j). \tag{32}$$

This leads to the following algorithm for the reconstruction of $f \in PW_\Omega^M$ from the finite collection of nonuniform samples $\{f(\lambda_l)\}_{l=-N}^N$:

$$f_0 = \frac{1}{1+\gamma^2} T_\Omega^M f; \qquad f_{j+1} = f_j + \frac{1}{1+\gamma^2} T_\Omega^M (f - f_j).$$

Again, the frame algorithm may be cast in terms of uniform samples $f(k/\Omega)$ ($|k| \le M$). In fact if $a_k^{(j)} = f_j(k/\Omega)$, then the frame iteration may be written

$$a_k^{(j+1)} = a_k^{(j)} + a_k^{(0)} - (Fa^{(j)})_k \qquad (|k| \le M,\ j \ge 0) \tag{33}$$

where F is the $(2M+1) \times (2M+1)$ matrix with (k,n)-th entry

$$F_{kn} = \frac{1}{1+\gamma^2} \sum_{l=-N}^{N} v_l \operatorname{sinc}(\Omega\lambda_l - n) \operatorname{sinc}(k - \Omega\lambda_l).$$

4.2 Complexity

To estimate the complexity of the algorithm, observe that

$$f_j(\lambda_l) = \sum_{k=-M}^{M} a_k^{(j)} \operatorname{sinc}(\Omega\lambda_l - k)$$

so that computation of $\{f_j(\lambda_l)\}_{l=-N}^N$ requires $O(MN)$ operations. Furthermore,

$$\begin{aligned} (Fa^{(j)})_k &= \frac{1}{1+\gamma^2} \sum_{n=-M}^{M} a_n^{(j)} \Big(\sum_{l=-N}^{N} v_l \operatorname{sinc}(n - \Omega\lambda_l) \operatorname{sinc}(k - \Omega\lambda_l) \Big) \\ &= \frac{1}{1+\gamma^2} \sum_{l=-N}^{N} v_l \operatorname{sinc}(k - \Omega\lambda_l) f_j(\lambda_l) \end{aligned}$$

so that the collection $\{(Fa^{(j)})_k\}_{k=-M}^M$ may be computed in $O(MN)$ operations. With

$$a_k^{(0)} = \frac{1}{1+\gamma^2} \sum_{l=-N}^{N} v_l f(\lambda_l) \operatorname{sinc}(k - \Omega\lambda_l),$$

the collection $\{a_k^{(0)}\}_{k=-M}^M$ may be computed in $O(MN)$ operations. Hence the frame iteration (33) has complexity $O(MN)$. The frame iterates f_j satisfy (15)

with $A = (1-\sqrt{\delta^2\Omega^2+\omega})^2$ and $B = (1+\sqrt{\delta^2\Omega^2+\omega})^2$. The algorithm is iterated until the J-th iterate approximates f with relative error ε, *i.e.*, $\|f - f_J\|_2^2 \leq \varepsilon\|f\|_2^2$. With B and A as above, this may be achieved with $J \geq \dfrac{\log(1/\varepsilon) + \log\kappa}{\log(1/\rho)}$. Now

$$\kappa = \frac{B}{A} = \frac{(1+\sqrt{\delta^2\Omega^2+\omega})^2}{(1-\sqrt{\delta^2\Omega^2+\omega})^2} \leq \frac{4}{(1-\sqrt{\delta^2\Omega^2+\omega})^2}$$

and

$$\frac{1}{\rho} = \frac{B-A}{B+A} = \frac{1+\delta^2\Omega^2+\omega}{2\sqrt{\delta^2\Omega^2+\omega}} = 1 + \frac{(1-\sqrt{\delta^2\Omega^2+\omega})^2}{2\sqrt{\delta^2\Omega^2+\omega}}$$

so that, since $\log(1+x) \geq (\log 2)x$ for $0 < x < 1$,

$$\log\left(\frac{1}{\rho}\right) \geq \log 2 \frac{(1-\sqrt{\delta^2\Omega^2+\omega})^2}{2\sqrt{\delta^2\Omega^2+\omega}} > \left(\frac{\log 2}{2}\right)(1-\sqrt{\delta^2\Omega^2+\omega}).$$

Therefore, it is sufficient to have

$$\begin{aligned} J &\geq \frac{2}{\log 2}\left(\frac{\log(1/\varepsilon) + \log 4 + 2\log(1/(1-\sqrt{\delta^2\Omega^2+\omega}))}{(1-\sqrt{\delta^2\Omega^2+\omega})^2}\right) \\ &= O\left(\frac{\log(1/\varepsilon)}{(1-\sqrt{\delta^2\Omega^2+\omega})^2}\right) \end{aligned}$$

provided $\varepsilon \ll 1 - \sqrt{\delta^2\Omega^2+\omega}$. Finally, the complexity of this algorithm in reconstructing $f \in PW_\Omega^M$ from nonuniform samples $\{f(\lambda_l)\}_{l=-N}^N$ with relative error ε is $O(MN\log(1/\varepsilon)(1-\sqrt{\delta^2\Omega^2+\omega})^{-2})$.

Since PW_Ω^M has dimension $2M+1$, the conjugate gradient frame algorithm [10], [15], each iteration of which is an $O(MN)$ operation, will reproduce f *exactly* after $2M+1$ iterations. Hence, this algorithm can be used to reproduce $f \in PW_\Omega^M$ without error with complexity $O(M^2N)$. Hence, the conjugate gradient frame algorithm becomes competitive (in terms of speed) when ε is small or $\delta\Omega$ is close to 1.

5 Nonuniform sampling of bandlimited, ε-timelimited signals

We now turn our attention to the *approximate* reconstruction of $f \in PW_\Omega$ from a finite collection of nonuniform samples $\{f(\lambda_l)\}_{l=-N}^{N}$. There are several approaches to this problem in the literature. Gröchenig [11] and Strohmer [18] use the samples of f to generate a best fit trigonometric polynomial, the coefficients of which may be computed by solving a Toeplitz system. Xiao, Rokhlin and Yarvin [22] and Beylkin and Monzón [3] take an entirely different approach (see also Hogan and Lakey [15] for a review of these ideas). By expanding complex exponentials in prolate spheroidal series, certain Gaussian quadrature properties of the prolate spheroidal wavefunctions may be exploited to obtain efficient sampling representations of bandlimited, ε-timelimted signals. The disadvantage of these methods is that the sampling points are very restrictive – in fact they are nodes of the aforementioned Gaussian quadrature scheme. On the other hand, the rapid decay of the eigenvalues associated with the prolate spheroidal wavefunctions give these sampling expansions good approximation properties, *i.e.*, rapid convergence with relatively few samples.

We take a different approach here, combining certain estimates of the local approximation power of sampling expansions in the spirit of Natterer [16] with the finite-dimensional version of the adaptive frame expansions of the previous section.

The first step is to find $T > 0$ for which $\int_{|t|>T} |f(t)|^2\,dt < \varepsilon \|f\|_2^2$. For this

value of T we say that f is ε-timelimited to the interval $[-T/2, T/2]$. Our samples will be chosen in and around this interval. The idea is to choose M large enough so that $\|f - P_\Omega^M f\|_2^2 < \varepsilon \|f\|_2^2$ and then approximate $P_\Omega^M f$ by the technique of the previous section.

Complexity estimates benefit from certain uncertainty considerations. If f is a signal that is bandlimited to the interval $[-\Omega/2, \Omega/2]$ and ε-timelimited to $[-T/2, T/2]$, then decreasing ε forces T to increase, as the following result, due to Nazarov [13], shows.

Theorem 7. *For all $f \in L^2(\mathbb{R})$, we have*

$$\|f\|_2^2 \le C e^{\alpha \Omega T} \left(\int_{|t|>T/2} |f(t)|^2 \, dt + \int_{|\xi|>\Omega/2} |\hat{f}(\xi)|^2 \, d\xi \right) \tag{34}$$

with α, C positive constants independent of f.

Let $\mathcal{F}(\Omega, T; \varepsilon)$ be the class of those $f \in PW_\Omega$ for which $\int_{|s|>T/2} |f(s)|^2 \, ds \le \varepsilon \|f\|_2^2$.

Corollary 1. *Let $\Omega T \ge 1$ and $f \in \mathcal{F}(\Omega, T; \varepsilon)$. Then $\Omega T \ge \dfrac{\log(1/\varepsilon)}{\alpha + \log C}$, where α, C are the universal constants of Theorem 7.*

Proof. Since $f \in \mathcal{F}(\Omega, T; \varepsilon)$, we have $\int_{|\xi|>\Omega/2} |\hat{f}|^2 = 0$ and $\int_{|t|>T/2} |f|^2 \le \varepsilon \|f\|_2^2$. Hence, with an application the (34), $\Omega T \ge \dfrac{1}{\alpha} \log\left(\dfrac{1}{C\varepsilon}\right)$. By choosing any nontrivial $f \in PW_\Omega$ and letting $T \to 0$ in (34), we find that $C \ge 1$. Let $F(\varepsilon) = \dfrac{1}{\alpha} \log\left(\dfrac{1}{C\varepsilon}\right)$ and note that $F(\varepsilon) \ge 1 \iff \varepsilon \le \dfrac{1}{Ce^\alpha}$. Let

$$G(\varepsilon) = \frac{F(\varepsilon)}{\log(1/\varepsilon)} = \frac{1}{\alpha} \left(\frac{\log C}{\log \varepsilon} + 1 \right) \qquad (\varepsilon > 0).$$

Then $\lim_{\varepsilon\to 0^+} G(\varepsilon) = \dfrac{1}{\alpha}$, $G\big(\dfrac{1}{Ce^{\alpha}}\big) = \dfrac{1}{\alpha+\log C}$ and $G'(\varepsilon) = \dfrac{-\log C}{\alpha\varepsilon(\log\varepsilon)^2} \le 0$ since $C \ge 1$. Hence $\dfrac{1}{\alpha+\log C} \le G(\varepsilon) \le \dfrac{1}{\alpha}$ for $0 < \varepsilon < \dfrac{1}{Ce^{\alpha}}$, *i.e.*, if $0 < \varepsilon < \dfrac{1}{Ce^{\alpha}}$, then $\Omega T \ge F(\varepsilon) = G(\varepsilon)\log\big(\dfrac{1}{\varepsilon}\big) \ge \dfrac{1}{\alpha+\log C}\log\big(\dfrac{1}{\varepsilon}\big)$. On the other hand, if $\varepsilon \ge \dfrac{1}{Ce^{\alpha}}$ then $\log\big(\dfrac{1}{\varepsilon}\big) \le \alpha + \log C$ and $\Omega T \ge 1 \ge \dfrac{1}{\alpha+\log C}\log\big(\dfrac{1}{\varepsilon}\big)$. Hence, for all $\varepsilon > 0$, $\Omega T \ge \dfrac{1}{\alpha+\log C}\log\big(\dfrac{1}{\varepsilon}\big)$. □

For the sake of the efficiency of the algorithm outlined in section 5.1, it is important that M be not much larger than $\Omega T/2$. The following result provides an upper bound on M with suitable asymptotics.

Lemma 1. *If* $1/2 \le \sigma < 1$ *and* $f \in \mathcal{F}(\Omega, T; \varepsilon)$*, then there exists* $M = O(\Omega T/2 + \log^{1/\sigma}(1/\varepsilon) + (1-\sigma)^{-1/\sigma})$ *such that* $\|f - P_{\Omega}^{M} f\|_2^2 \le c\varepsilon\|f\|_2^2$ *with* c *independent of* f, Ω *and* M.

Proof. Let φ_Ω satisfy (2) with $\lambda = 2$ and the subexponential decay estimate (6). Since $\widehat{\varphi_\Omega} \equiv 1$ on $[-\Omega/2, \Omega/2]$, we have $f(t) = \int f(s)\varphi_\Omega(t-s)\,ds$, so that with $\mu \ge 0$ and $M = \mu + \Omega T/2$,

$$\begin{aligned}
\frac{1}{\Omega}\sum_{|k|>M} |f(k/\Omega)|^2 &= \frac{1}{\Omega}\sum_{|k|>M}\left|\int f(s)\varphi_\Omega\Big(\frac{k}{\Omega}-s\Big)\,ds\right|^2 \\
&\le \frac{2}{\Omega}\sum_{|k|>M}\left|\int_{|s\le T/2} f(s)\varphi_\Omega\Big(\frac{k}{\Omega}-s\Big)\,ds\right|^2 \\
&\quad + \frac{2}{\Omega}\sum_{|k|>M}\left|\int_{|s>T/2} f(s)\varphi_\Omega\Big(\frac{k}{\Omega}-s\Big)\,ds\right|^2 = K + L.
\end{aligned}$$

However, by the subexponential decay of φ_Ω and the assumption that $\sigma \ge 1/2$,

an application of the Cauchy-Schwarz inequality gives,

$$\begin{aligned}
K &\le C2^{2/(1-\sigma)}\Omega \sum_{|k|>M}\left(\int_{|s|\le T/2}|f(s)|e^{-|\pi(k-\Omega s)|^\sigma}\,ds\right)^2\\
&\le C2^{2/(1-\sigma)}\Omega\|f\|_2^2\sum_{|k|>M}\int_{-\Omega T/2-k}^{\Omega T/2-k}e^{-2|\pi s|^\sigma}\,ds\\
&\le C2^{2/(1-\sigma)}\|f\|_2^2\sum_{k=M+1}^{\infty}\int_{(k-\Omega T/2)^\sigma}^{\infty}e^{-\pi^\sigma s}\,ds\\
&= C2^{2/(1-\sigma)}\|f\|_2^2\sum_{k=M+1}^{\infty}e^{-\pi^\sigma(k-\Omega T/2)^\sigma}\\
&\le C2^{2/(1-\sigma)}\|f\|_2^2\int_{\mu\sigma}^{\infty}e^{-(\pi y)^\sigma}\,dy\\
&\le C2^{2/(1-\sigma)}\|f\|_2^2e^{-\pi^\sigma(M-\Omega T/2)^\sigma/2}<\varepsilon\|f\|_2^2
\end{aligned}$$

provided $M > \dfrac{\Omega T}{2}+\dfrac{2^{1/\sigma}}{\pi}\log^{1/\sigma}\left(\dfrac{2^{2/(1-\sigma)}C}{\varepsilon}\right)$, *i.e.*,

$$M = O\left(\frac{\Omega T}{2}+\log^{1/\sigma}\left(\frac{1}{\varepsilon}\right)+\frac{1}{(1-\sigma)^{1/\sigma}}\right).$$

On the other hand, integral L may be estimated by

$$L\le\frac{2}{\Omega}\sum_{k=-\infty}^{\infty}\left|(f\chi_{|s|>T/2})*\varphi_\Omega\left(\frac{k}{\Omega}\right)\right|^2.$$

However, $(f\chi_{|s|>T/2}*\varphi_\Omega)\hat{}=(f\chi_{|s|>T/2})\hat{}\,\widehat{\varphi_\Omega}=0$ for $|\xi|>\Omega$. Therefore, since $|\widehat{\varphi_\Omega}(\xi)|\le 1$,

$$\begin{aligned}
L &\le \frac{4}{2\Omega}\sum_{k=-\infty}^{\infty}\left|(f\chi_{|s|>T/2})*\varphi_\Omega\left(\frac{k}{2\Omega}\right)\right|^2\\
&= 4\|(f\chi_{|s|>T/2})*\varphi_\Omega\|_2^2=4\|(f\chi_{|s|>T/2})\hat{}\,\widehat{\varphi_\Omega}\|_2^2\le 4\|f\chi_{|s|>T/2}\|_2^2\le 4\varepsilon\|f\|_2^2,
\end{aligned}$$

since $f\in\mathcal{F}(\Omega,T;\varepsilon)$. Consequently, the energy of the samples $\{f(k/\Omega)\}$ is localized near the interval $[-T/2,T/2]$ in the sense that for some $M=O\left(\dfrac{\Omega T}{2}+\log^{1/\sigma}\left(\dfrac{1}{\varepsilon}\right)+\dfrac{1}{(1-\sigma)^{1/\sigma}}\right)$ we have the inequality

$$\frac{1}{\Omega}\sum_{|k|>M}|f(k/\Omega)|^2\le c\varepsilon\|f\|_2^2.$$

□

The data is a collection of samples $\{f(\lambda_l)\}_{l=-N}^{N}$ obtained from a signal $f \in \mathcal{F}(\Omega, T; \varepsilon)$. We want to reconstruct f *approximately* from these samples in an efficient manner. Before describing the algorithm, we review a few technical results.

Lemma 2. *Let $\Lambda_N = \{\lambda_l\}_{l=-N}^{N}$ be as in Theorem 6 and $v_l = (\lambda_{l+1} - \lambda_{l-1})/2$ ($|l| \le N-1$). If $f \in PW_\Omega$ then $\sum_{l=-N}^{N} v_l |f(\lambda_l)|^2 \le (1+\delta\Omega)^2 \|f\|_2^2$.*

Proof. In the proof of Theorem 6 we saw that Wirtinger's inequality (19) implies that when $Q_{\Lambda_N} f = \sum_{l=-N}^{N} f(\lambda_l)\chi_{[\eta_l, \eta_{l+1}]}$,

$$\int_{\lambda_{-N}}^{\lambda_N} |f(t) - Q_{\Lambda_N} f(t)|^2 \, dt \le \delta^2 \Omega^2 \|f\|_2^2.$$

Furthermore, $\|Q_{\Lambda_N} f\|_2^2 = \sum_{l=-N}^{N} v_l |f(\lambda_l)|^2$ so that

$$\left(\sum_{l=-N}^{N} v_l |f(\lambda_l)|^2 \right)^{1/2} = \|Q_{\Lambda_N} f\|_2 \le \|f\|_2 + \|f - Q_{\Lambda_N} f\|_2 \le (1+\delta\Omega)\|f\|_2.$$

□

As a corollary to Lemmas 1 and 2, we have

Corollary 2. *Let Λ_N, v_l and M be as in Lemmas 1 and 2. Then if $f \in PW_\Omega$,*

$$\sum_{l=-N}^{N} v_l |f(\lambda_l) - P_\Omega^M f(\lambda_l)|^2 \le (1+\delta\Omega)^2 \|f - P_\Omega^M f\|_2^2 \le c\varepsilon (1+\delta\Omega)^2 \|f\|_2^2.$$

Lemma 3. *Let $\Lambda_N = \{\lambda_l\}_{l=-N}^{N}$, v_l ($|l| \le N$) be as in Theorem 6 and let G be the $(2M+1) \times (2N+1)$ matrix with (j,l)-th entry $G_{jl} = \sqrt{v_l}\, sinc\,(j - \Omega\lambda_l)$ ($|j| \le M$, $|l| \le N$) acting as a linear operator from $\mathbb{C}^{2N+1}$ to $\mathbb{C}^{2M+1}$ by matrix multiplication. Then G has operator norm $\|G\| \le 2/\sqrt{\Omega}$.*

Proof. Let $a \in \mathbb{C}^{2M+1}$ and define $g \in PW_\Omega^M$ by $g(t) = \sum_{j=-M}^{M} a_j \operatorname{sinc}(\Omega t - j)$ so that $g(k/\Omega) = a_k$ ($|k| \le M$). Then with G^* the Hermitian adjoint of G, we have

$$(G^*a)_l = \sum_{j=-M}^{M} B_{lj}^* a_j = \sum_{j=-M}^{M} \sqrt{v_l} \operatorname{sinc}(j - \Omega\lambda_l) g(j/\Omega) = \sqrt{v_l} g(\lambda_l)$$

so that by Lemma 2,

$$\|G^*a\|_{\ell^2}^2 = \sum_{l=-N}^{N} v_l |g(\lambda_l)|^2 \le (1+\delta\Omega)^2 \|g\|_2^2 \le \frac{4}{\Omega}\|a\|_{\ell^2}^2,$$

i.e., $\|G^*\| \le 2/\sqrt{\Omega}$ and hence $\|G\| \le 2/\sqrt{\Omega}$. □

Lemma 4. *Let Λ_N, v_l, φ_l^M be as in Theorem 6 and $\gamma < 1$ be as in (25). Let F be the $(2M+1) \times (2M+1)$ matrix with (k,j)-th entry*

$$F_{kj} = \frac{1}{1+\gamma^2} \sum_{l=-N}^{N} v_l \, sinc(\Omega\lambda_l - j) \, sinc(k - \Omega\lambda_l) \qquad (|j|, |k| \le M). \tag{35}$$

Then $\|I - F\| \le \dfrac{2\gamma}{1+\gamma^2} < 1$.

Proof. Given $b \in \mathbb{C}^{2M+1}$, define $g \in PW_\Omega^M$ by $g(t) = \sum_{j=-M}^{M} b_j \operatorname{sinc}(\Omega t - j)$. Then

$$\begin{aligned} b_k - (Fb)_k &= g\left(\frac{k}{\Omega}\right) - \sum_{n=-M}^{M} F_{kn} g\left(\frac{n}{\Omega}\right) \\ &= g\left(\frac{k}{\Omega}\right) - \frac{1}{1+\gamma^2} \sum_{n=-M}^{M} \sum_{l=-N}^{N} v_l \operatorname{sinc}(\Omega\lambda_l - n) \operatorname{sinc}(k - \Omega\lambda_l) g\left(\frac{n}{\Omega}\right) \\ &= g\left(\frac{k}{\Omega}\right) - \frac{1}{1+\gamma^2} \sum_{l=-N}^{N} v_l g(\lambda_l) \operatorname{sinc}(k - \Omega\lambda_l) \\ &= \left(\left(I - \frac{1}{1+\gamma^2} T_\Omega^M\right) g\right)\left(\frac{k}{\Omega}\right), \end{aligned}$$

where T_Ω^M is the frame operator associated with the frame $\{\varphi_l^M\}_{l=-N}^N$ of Theorem 6. Since $g \in PW_\Omega^M$ we may use estimate (11) to obtain

$$\begin{aligned}\|b - Fb\|_{\ell^2}^2 &= \sum_{k=-M}^{M} \left| \left(I - \frac{1}{1+\gamma^2} T_\Omega^M \right) g\left(\frac{k}{\Omega}\right) \right|^2 \\ &= \Omega \int \left| \left(I - \frac{1}{1+\gamma^2} T_\Omega^M \right) g(t) \right|^2 dt \leq \left(\frac{2\gamma}{1+\gamma^2} \right)^2 \|b\|_{\ell^2}^2.\end{aligned}$$

□

We are now in a position to describe an algorithm that treats signals in $\mathcal{F}(\Omega, T; \varepsilon)$. Given $f \in \mathcal{F}(\Omega, T; \varepsilon)$, choose M such that $\|f - P_\Omega^M f\|_2^2 \leq \varepsilon \|f\|_2^2$. By Lemma 1, for each $0 < \sigma < 1$, M may be chosen to be $O(\Omega T/2 + \log^{1/\sigma}(1/\varepsilon) + (1-\sigma)^{-1/\sigma})$. Let $\Lambda_N = \{\lambda_l\}_{l=-N}^N$ be a sampling set satisfying the conditions of Theorem 6. If the nonuniform samples $P_\Omega^M f(\lambda_l)$ ($|l| \leq N$) were fed to the algorithm of section 4 ($f_0 = \frac{1}{1+\gamma^2} T_\Omega^M P_\Omega^M f$) then we could reconstruct the uniform samples $P_\Omega^M f(j/\Omega)$ ($|j| \leq M$) and hence recover $P_\Omega^M f$. However, we only have at our disposal the samples $f(\lambda_l)$ ($|l| \leq N$). By Corollary 2, the (weighted) ℓ^2-error between these nonuniform samples is

$$\sum_{l=-N}^{N} v_l |f(\lambda_l) - P_\Omega^M f(\lambda_l)|^2 \leq c\varepsilon \|f\|_2^2.$$

Let $a^{(0)}, \tilde{a}^{(0)} \in \mathbb{C}^{2M+1}$ be given by

$$\begin{aligned} a_k^{(0)} &= \frac{1}{1+\gamma^2} \sum_{l=-N}^{N} v_l f(\lambda_l) \operatorname{sinc}(k - \Omega\lambda_l); \\ \tilde{a}_k^{(0)} &= \frac{1}{1+\gamma^2} \sum_{l=-N}^{N} v_l P_\Omega^M f(\lambda_l) \operatorname{sinc}(k - \Omega\lambda_l). \end{aligned}$$

Iterates $a^{(j)}$, $\tilde{a}^{(j)}$ are defined by

$$a^{(j)} = H a^{(j-1)} + a^{(0)}; \quad \tilde{a}^{(j)} = H \tilde{a}^{(j-1)} + \tilde{a}^{(0)}$$

with $H = I - F$ and F as in (35). We want to estimate the difference between the j-th iterates $a^{(j)}$ and $\tilde{a}^{(j)}$.

Lemma 5. *Let $f \in \mathcal{F}(\Omega, T; \varepsilon)$, $a^{(j)}, \tilde{a}^{(j)}$ be as above, $\gamma < 1$ be as in (25) and $\varepsilon_j = \|a^{(j)} - \tilde{a}^{(j)}\|_{\ell^2}$. Then*

$$\varepsilon_j \leq \frac{1+\gamma^2}{(1-\gamma)^2}\varepsilon_0 \leq c\sqrt{\varepsilon}\|f\|_2. \tag{36}$$

Proof. First note that with G the $(2M+1) \times (2N+1)$ matrix of Lemma 3,

$$\begin{aligned}
\|a^{(0)} - \tilde{a}^{(0)}\|_{\ell^2} &= \frac{1}{1+\gamma^2}\Bigg(\sum_{k=-M}^{M} \Bigg| \sum_{l=-N}^{N} v_l (f(\lambda_l) - P_\Omega^M f(\lambda_l)) \operatorname{sinc}(k - \Omega\lambda_l) \Bigg|^2 \Bigg)^{1/2} \\
&= \frac{1}{1+\gamma^2}\Bigg(\sum_{k=-M}^{M} \Bigg| \sum_{l=-N}^{N} G_{kl} (\sqrt{v_l}(f(\lambda_l) - P_\Omega^M f(\lambda_l))) \Bigg|^2 \Bigg)^{1/2} \\
&\leq \frac{1}{1+\gamma^2}\frac{2}{\sqrt{\Omega}}\Bigg(\sum_{l=-N}^{N} v_l |f(\lambda_l) - P_\Omega^M f(\lambda_l)|^2 \Bigg)^{1/2} \leq c\sqrt{\varepsilon}\|f\|_2
\end{aligned}$$

where the penultimate inequality is an application of Lemma 3 and the last inequality is an application of Corollary 2. Furthermore, $a^{(j)} - \tilde{a}^{(j)} = H(a^{(j-1)} - \tilde{a}^{(j-1)}) + a^{(0)} - \tilde{a}^{(0)}$, so that by Corollary 2 and Lemma 4

$$\begin{aligned}
\varepsilon_j &\leq \|H\|\varepsilon_{j-1} + \varepsilon_0 \\
&\leq \frac{2\gamma}{1+\gamma^2}\varepsilon_{j-1} + \varepsilon_0 \\
&\leq \frac{2\gamma}{1+\gamma^2}\left(\frac{2\gamma}{1+\gamma^2}\varepsilon_{j-2} + \varepsilon_0 \right) + \varepsilon_0 \\
&\leq \sum_{l=0}^{j-1} \left(\frac{2\gamma}{1+\gamma^2} \right)^l \varepsilon_0 \leq \frac{1+\gamma^2}{(1-\gamma)^2}\varepsilon_0 \leq c\sqrt{\varepsilon}\|f\|_2.
\end{aligned}$$

□

We see then that at each stage of the finite-dimensional algorithm, the error in the j-th iterate ε_j is controlled by the error in the initialization ε_0.

The sequence $a^{(j)}$ converges geometrically to $a \in \mathbb{C}^{2M+1}$. In fact we have

Lemma 6. *Let $f \in \mathcal{F}(\Omega, T; \varepsilon)$ and $a^{(j)}, a \in \mathbb{C}^{2M+1}$ as above. Then*

$$\|a^{(j)} - a\|_{\ell^2} \leq \frac{c\sqrt{\varepsilon}}{\sqrt{\Omega}} \|f\|_2$$

with

$$c = \frac{8\gamma}{(1-\gamma)^2(1+\gamma^2)}, \quad \textit{and} \quad j \geq \frac{\log \varepsilon}{2\log(2\gamma/(1+\gamma^2))}.$$

Proof. Let F be the $(2M+1) \times (2M+1)$ matrix of Lemma 4 and $H = I - F$. Since $a^{(j)} = \sum_{l=0}^{j} H^l a^{(0)}$, we have

$$\begin{aligned} \|a - a^{(j)}\|_{\ell^2} &\leq \sum_{l=j+1}^{\infty} \|H\|^l \|a^{(0)}\|_{\ell^2} \\ &= \frac{\|H\|^{j+1}}{1 - \|H\|} \|a^{(0)}\|_{\ell^2} \leq \left(\frac{2\gamma}{1+\gamma^2}\right)^j \frac{2\gamma}{(1-\gamma)^2} \|a^{(0)}\|_{\ell^2}. \end{aligned} \quad (37)$$

However, $a^{(0)} = \dfrac{1}{1+\gamma^2} Gc$ with G the $(2M+1) \times (2N+1)$ matrix of Lemma 3 and $c_l = \sqrt{v_l} f(\lambda_l)$ $(|l| \leq N)$, so by Lemmas 2 and 3,

$$\|a^{(0)}\|_{\ell^2} \leq \frac{1}{1+\gamma^2} \|G\| \left(\sum_{l=-N}^{N} v_l |f(\lambda_l)|^2 \right)^{1/2} \leq \frac{4}{(1+\gamma^2)\sqrt{\Omega}} \|f\|_2. \quad (38)$$

Combining (37) and (38) gives the result. □

Since, in general, $f \notin PW_\Omega^M$, the iterates $a_k^{(j)}$ cannot be expected to converge to $b_k = f(k/\Omega)$. However, we have the following error estimate:

Proposition 1. *Suppose $f \in \mathcal{F}(\Omega, T; \varepsilon)$ and $b \in \mathbb{C}^{2M+1}$ is given by $b_k = f(k/\Omega)$ $(|k| \leq M)$ and $a^{(0)}, a^{(j)}$ are as above. Then $a^{(j)} \to a$ and $\|b - a\|_{\ell^2}^2 \leq c\varepsilon\Omega \|f\|_2^2$.*

Proof. Let $\tilde{a}^{(j)}$ be as above and $\tilde{a}^{(j)} \to \tilde{a}$. Then

$$\|b - a\|_{\ell^2} \leq \|b - \tilde{a}\|_{\ell^2} + \|\tilde{a} - \tilde{a}^{(j)}\|_{\ell^2} + \|\tilde{a}^{(j)} - a^{(j)}\|_{\ell^2} + \|a^{(j)} - a\|_{\ell^2}. \quad (39)$$

However, the first term on the right hand side may be bounded by

$$\|b - \tilde{a}\|_2^2 = \sum_{k=-M}^{M} \left| f\left(\frac{k}{\Omega}\right) - P_\Omega^M f\left(\frac{k}{\Omega}\right) \right|^2 = \Omega \|f - P_M^\Omega f\|_2^2 \leq c\Omega\varepsilon \|f\|_2^2$$

by Lemma 1. Furthermore, $\tilde{a}^{(j)} \to \tilde{a}$ geometrically:

$$\begin{aligned} \|\tilde{a} - \tilde{a}^{(j)}\|_{\ell^2} &\leq c\gamma^{2j+2} \sum_{k=-M}^{M} |P_\Omega^M f(k/\Omega)|^2 \\ &= c\Omega\gamma^{2j+2} \|P_\Omega^M f\|_2^2 \leq c\Omega\gamma^{2j+2} \|f\|_2^2 \to 0 \end{aligned}$$

as $j \to \infty$. Also, by (36), $\|a^{(j)} - \tilde{a}^{(j)}\|_{\ell^2}^2 \leq c\varepsilon \|f\|_2^2$. Finally, $\|a - a^{(j)}\|_{\ell^2} \to 0$ as $j \to \infty$ by Lemma 6. Combining these estimates with (39) gives the result. □

Finally we have:

Proposition 2. *Let $f \in \mathcal{F}(\Omega, T; \varepsilon)$ and $a^{(j)}, b \in \mathbb{C}^{2M+1}$ be as above. Then*

$$\|b - a^{(j)}\|_{\ell^2}^2 \leq c\varepsilon\Omega \|f\|_2^2$$

for $j = O(\log 1/\varepsilon)$.

Proof. By Lemma 6 and Proposition 1 we have

$$\|b - a^{(j)}\|_{\ell^2} \leq \|b - a\|_{\ell^2} + \|a - a^{(j)}\|_{\ell^2} \leq c\sqrt{\varepsilon\Omega} \|f\|_2 + c\sqrt{\frac{\varepsilon}{\Omega}} \|f\|_2.$$

□

5.1 Algorithm

In summary, the algorithm for the approximate reconstruction of $f \in PW_\Omega$ from nonuniform samples $\{f(\lambda_l)\}_{l=-N}^{N}$ proceeds as follows:

Step 1. Let Λ be a doubly infinite sequence with density δ as in Theorem 6. Choose T such that $f \in \mathcal{F}(\Omega, T; \varepsilon)$, $0 < \sigma < 1$ and $M = O(\Omega T/2 + \log^{1/\sigma}(1/\varepsilon) + (1-\sigma)^{-1/\sigma})$ such that $\|f - P_\Omega^M f\|_2^2 < \varepsilon \|f\|_2^2$ as in Lemma 1. Choose $N = O((M/\omega + (1-\sigma)^{-1/\sigma})/(\delta\Omega))$ such that if γ is defined as in (25), then $\gamma < 1$.

Step 2. Compute $a^{(0)} \in \mathbb{C}^{2M+1}$ by

$$a_k^{(0)} = \frac{1}{1+\gamma^2} \sum_{l=-N}^{N} v_l f(\lambda_l) \operatorname{sinc}(k - \Omega\lambda_l) \qquad (|k| \le M).$$

Step 3. Compute the iterates $a^{(j)} \in \mathbb{C}^{2M+1}$ defined by

$$a^{(j)} = H a^{(j-1)} + a^{(0)} \tag{40}$$

with $H = I - F$ and F as in Lemma 4 by splitting the process into two steps. Define first a vector $w^{(j)} \in \mathbb{C}^{2N+1}$ by

$$w_l^{(j)} = \sum_{n=-N}^{N} a_n^{(j)} \operatorname{sinc}(n - \Omega\lambda_l),$$

an $O(MN)$ operation. Now

$$(Fa^{(j)})_k = \frac{1}{1+\gamma^2} \sum_{l=-N}^{N} v_l \operatorname{sinc}(k - \Omega\lambda_l) w_l^{(j)},$$

another $O(MN)$ operation. Hence the iteration (40) is an $O(MN)$ operation. Notice that we avoided computing the entries of F which would require $O(MN^2)$ operations.

Step 4. Stop the iteration of Step 3 after

$$J = \frac{2}{\log 2} \left[\frac{\log(1/\varepsilon) + \log 4 + 2\log(1/(1 - \sqrt{\delta^2\Omega^2 + \omega}))}{(1 - \sqrt{\delta^2\Omega^2 + \omega})^2} \right] + 1$$

iterations. Here $[x]$ is the largest integer $n \leq x$. The vector $a^{(J)}$ is our approximation of the desired vector $b \in \mathbb{C}^{2M+1}$ of uniform samples of f, *i.e.*, with $b_k = f(k/\Omega)$ $(|k| \leq M)$ we have $\|b - a^{(J)}\|_2^2 \leq c\varepsilon\Omega\|f\|_2^2$.

5.2 Complexity

To estimate the total complexity of the algorithm, notice that each iterate of the finite-dimensional algorithm requires $O(MN)$ operations and we have determined that we require $O(\log(1/\varepsilon)(1 - \sqrt{\delta^2\Omega^2 + \omega})^{-2})$ iterations to obtain the desired accuracy. Hence the algorithm has complexity $O(MN\log(1/\varepsilon)(1 - \sqrt{\delta^2\Omega^2 + \omega})^{-2})$. However $M = O(\Omega T/2 + \log^{1/\sigma}(1/\varepsilon) + (1-\sigma)^{-1/\sigma})$ and $N = O(M/(\omega\delta\Omega))$, so that in terms of Ω, T, ε, σ and ω, the algorithm has complexity

$$O\bigg(\frac{M^2\log(1/\varepsilon)}{\omega\delta\Omega(1-\sqrt{\delta^2\Omega^2+\omega})^2}\bigg) = O\bigg(\bigg(\frac{\Omega^2T^2 + \log^{2/\sigma}(1/\varepsilon) + (1-\sigma)^{-2/\sigma}}{\omega\delta\Omega(1-\sqrt{\delta^2\Omega^2+\omega})^2}\bigg)\log(1/\varepsilon)\bigg)$$

provided $\varepsilon \ll 1 - \sqrt{\delta^2\Omega^2 + \omega}$. At this stage we can consider $\sigma \in [1/2, 1)$ as a *tuning parameter* – we are at liberty to choose σ arbitrarily in this range and we do so in an effort to simplify the statement of the complexity of the algorithm. In particular, if $\sigma = 1 - \log^{-1}(1/\varepsilon)$, then $(1-\sigma)^{-2/\sigma} = \log^{2/\sigma}(1/\varepsilon)$ and we see that for $\varepsilon \ll 1 - \sqrt{\delta^2\Omega^2 + \omega}$, the algorithmic complexity simplifies to

$$O\bigg(\bigg(\frac{\Omega^2T^2 + \log^{2+1/(2(\log(1/\varepsilon)-1))}(1/\varepsilon)}{\omega\delta\Omega(1-\sqrt{\delta^2\Omega^2+\omega})^2}\bigg)\log(1/\varepsilon)\bigg).$$

Furthermore, since $\Omega T \geq C\log(1/\varepsilon)$ by Corollary 1, the expression simplifies further to $O\bigg(\dfrac{(\Omega T)^{2+1/(2\log(1/\varepsilon)-1)}\log(1/\varepsilon)}{\omega\delta\Omega(1-\sqrt{\delta^2\Omega^2+\omega})^2}\bigg)$.

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Preface B

Part B contains a selection of papers partly associated with the special session on wavelets, other multiscale methods and their applications at the 995th AMS Meeting in Athens, Ohio, March 26-27, 2004, and the special session on the multiscale methods and sampling in time-frequency analysis at the 1000th AMS Meeting in Albuquerque, New Mexico, October 16-17, 2004.

The papers cover topics such as biorthogonal wavelet basis, multiresolution frames in higher dimensions, multiscale approximation in integral equations, and deconvolution and sampling problems.

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THE CONSTRUCTION OF d-DIMENSIONAL MULTIRESOLUTION ANALYSIS FRAMES

JOHN J. BENEDETTO AND JUAN R. ROMERO

Abstract. We extend some of the classical results of the theory of multiresolution analysis (MRA) frames to Euclidean space $\mathbb{R}^d, d > 1$, and provide relevant examples. In the process, we use the theory of shift-invariant subspaces to bring new insights to the theory of frame multiresolution analysis.

1. Introduction

1.1. **Background.** Frames were introduced in the 1950s to deal with problems in nonharmonic Fourier series [20]. They are an appropiate tool to deal with problems where redundancy, robustness, oversampling, and/or nonuniform sampling play a role. This paper deals with wavelet frames. There are different techniques to construct wavelet frames such as Frame Multiresolution Analysis (FMRA), Generalized Frame Multiresolution Analysis (GFMRA), Generalized Multiresolution Analysis (GMRA), the Unitary Extension Principle (UEP), and Mixed and Oblique Extension Principles, e.g., [2, 3, 6, 7, 8, 9, 19, 24, 25, 32, 33, 34, 36, 37, 38].

The simplest and first of these construction techniques deals with FMRAs. The present work was initiated by an attempt to extend to $\mathbb{R}^d, d > 1$, a measure theoretic *characterization* and accompanying *construction* of FMRA wavelet frames for $L^2(\mathbb{R})$ found in [9]. This measure theoretic point of view appeared independently in [27]. It fits in naturally with the theory of FMRA, which, in turn, gives rise to the construction of a multidimensional Mallat-Meyer algorithm for multiresolution analysis (MRA) wavelet frames by tensor products [5, 29, 18]. The Mallat-Meyer algorithm is fundamental in several aspects of signal processing. As such, a feature that makes FMRAs potentially useful in signal processing is the fact that the perfect reconstruction filter banks associated to them can be narrow band. Hence, FMRA filter banks provide a method and algorithm to obtain quantization noise reduction simultaneously with reconstruction of a given narrow-band signal [8, 9].

The aforementioned measure theoretic material and related algorithm are the subject of Sections 5 and 6. The results are incomplete and the work is ongoing.

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However, in the process, we developed more definitive and quantitative results in the theory of multidimensional FMRA.

1.2. **Outline.** Section 2 gives some definitions and preliminary results. In particular, the notions stated in this subsection are defined in Section 2. In Section 3 we discuss shift-invariant subspace theory.

In Sections 4 and 5 we generalize the theory of FMRA on $\mathbb{R}$ that were proved in [8, 9]. Our main results are Theorems 4 - 8 in Section 4. Theorem 9 in Section 5 is essential theoretically, but it is not sufficiently constructive in its present form for genuine implementation. Section 6 is devoted to the construction of wavelet frames for $L^2\left(\mathbb{R}^d\right)$ in the spirit of the Mallat-Meyer algorithm.

Theorems 4 - 8 provide the equations neccesary to state quantitative sufficient conditions in order that an FMRA should give rise to a wavelet frame for $L^2\left(\mathbb{R}^d\right)$. In fact, Theorem 7, which summarizes Theorems 4 - 6, gives sufficient conditions for translates of a given finite set of functions to be a wavelet frame for a basic subspace W_0 of $L^2\left(\mathbb{R}^d\right)$. W_0 is defined below. Theorem 8 summarizes all these equations in a matrix equation, much in the spirit of the Unitary Extension Principle (UEP). Theorem 5 and Theorem 8 provide the equations neccesary to state quantitative conditions in order that an FMRA should give rise to a wavelet frame for $L^2(\mathbb{R}^d)$.

Theorems 4 - 8 can be viewed as the basis of a fast d-dimensional wavelet algorithm resulting from dyadic FMRAs. In fact, these theorems show how two classes of filters defining such an algorithm arise. The first class, the *analysis* or *decomposition* filters, is used to analyze a given signal in various subbands. The second class, the *synthesis* or *reconstruction* filters, is used for the exact reconstruction of a decomposed signal. In this context,the main theorem of the paper is Theorem 6, which shows that the proposed fast wavelet decomposition algorithm is reversed by an exact reconstruction algorithm. Furthermore, the filters in Theorem 6 can be non-separable, meaning that they do not necessarily arise as linear combinations of the one dimensional tensor products of their respective filters. Moreover, the reconstruction filters are not necessarily the canonical duals of the decomposition filters in the sense of Li and Ogawa [30, 31]. This provide additional freedom in the filter design.

Theorem 9 of Section 5 was proved independently by Benedetto and Treiber [9] and Kim and Lim [26, 27]. It states that a neccesary and sufficient condition to obtain an optimal wavelet frame (i.e., the number of wavelets required to construct a wavelet frame for $L^2(\mathbb{R}^d)$ is no more than $2^d - 1$) is that a set related to the spectrum of the core subspace V_0 of the FMRA should have measure zero. This theorem also gives a lower bound for the number of wavelet frame wavelets in the worst case scenario, that is, when the set related to the spectrum of the core subspace V_0 of the FMRA has positive measure.

Although the reader can go directly to Section 2, the remainder of the Introduction is devoted to perspective dealing with GFMRA and UEP.

1.3. **GFMRA and GMRA.** A collection $\{V_j\}_{j=-\infty}^{\infty}$ of closed subspaces of $L^2\left(\mathbb{R}^d\right)$ is called a *Generalized Multiresolution Analysis* [2, 3] if the following conditions are satisfied:

- $V_j \subset V_{j+1}$, for all $j \in Z$.
- $D\left(V_j\right) = V_{j+1}$.
- $\overline{\cup_j V_j} = L^2(\mathbb{R}^d)$ and $\cap_j V_j = \{0\}$.

- V_0 is invariant under the lattice Z^d, i.e., $\tau_n V_0 \subset V_0$ for all $n \in Z^d$, or equivalently, V_0 is a shift-invariant subspace.

We shall have defined a *Generalized Frame Multiresolution Analysis* if we change the last condition to the following one: "there is a set $\{\varphi_i : i \in I\} \subset L^2\left(\mathbb{R}^d\right)$, where I is at most countable, such that $\left\{\tau_n \varphi_i : i \in I, n \in \mathbb{Z}^d\right\}$ is a frame for V_0" [33, 34]. Weber has proved that GFMRAs and GMRAs are essentially the same if one is interested in semiorthogonal wavelet frames [40]. FMRAs are GFMRAs generated by a single scaling function.

Since the core subspace V_0 of a GMRA is shift-invariant, the action of the translation group induced by $\mathbb{Z}^d$ on V_0 can be seen as a unitary representation π of V_0. Following [1], there exists a projection-valued measure P on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, and π is the Fourier transform of this projection-valued measure, that is,

$$\pi_n = \int_{\mathbb{T}^d} e^{2\pi i n\gamma} dP\left(\gamma\right),\ n \in \mathbb{Z}^d.$$

This operator-valued measure is uniquely determined by a nonnegative integer-valued function m, defined on $\mathbb{T}^d$, which is called the *multiplicity function*. The multiplicity function m plays the role of a scaling set of vectors in this context, because it encodes all the structural information about V_0 that a set of scaling vectors can provide [2, 3]. More specifically, the multiplicity function roughly counts how many times each character in $\mathbb{T}^d$ occurs in π. Using the information the multiplicity function provides about π, we form the direct sum

$$L^2\left(S_1\right) \bigoplus L^2\left(S_2\right) \bigoplus ...,$$

where

$$S_j = \left\{\gamma \in \mathbb{T}^d : m\left(\gamma\right) \geq j\right\}.$$

The properties of m guarantee the existence of a unitary map

$$J : V_0 \longrightarrow \bigoplus_{j=1}^{\infty} L^2\left(S_j\right)$$

which intertwines translation with modulation, that is,

$$\forall f \in V_0, \forall k \in \mathbb{Z}^d,\ Jf\left(\cdot + k\right) = e^{2\pi i k\cdot} Jf.$$

Setting $\varphi_j = J^{-1}\left(\left(..., \mathbf{1}_{S_j}, 0, ...\right)\right)$, the set $\left\{\varphi_j\left(\cdot + k\right) : j \geq 1, k \in \mathbb{Z}^d\right\}$ is a Parseval frame for V_0 [2, 3].

In the GFMRA theory, the starting point is a set of scaling vectors [33, 34]. The corresponding low-pass filter is a bounded operator-valued function. One recipe for a semiorthogonal wavelet set of vectors is therein proposed by means of a generalized Mallat-Meyer algorithm.

Baggett et al. [2] give an algorithm to construct scaling functions and wavelet frames from what they call generalized conjugate mirror filters in $L^2(\mathbb{R}^d)$ [2, 4]. In particular, they are able to construct a non-MRA wavelet in $L^2\left(\mathbb{R}\right)$ whose Fourier transform is C^∞ in an arbitary large interval [2]. The starting point for the construction of their generalized conjugate mirror filters is the prescribed multiplicity function. The scaling functions they construct form a minimal set of generators for the core subspace V_0.

Weber [39, 40] developed a method to prove the existence of wavelet frames associated with GFMRAs by proving an extension of Robertson's wandering subspace

theorem. His construction starts from scaling vectors, and, using spectral multiplicity, he provides an alternate functional analytic method, bypassing the step of producing low-pass filters.

1.4. **UEP.** Ron and Shen followed a different path [37, 38]. They start with a refinable function $\psi_0 \in L^2(\mathbb{R})$ with essentially bounded periodization $\Psi_0(\gamma) = \sum \left|\widehat{\psi_0}(\gamma+k)\right|^2$ such that

$$\lim_{\gamma \longrightarrow 0} \widehat{\psi_0}(\gamma) = 1.$$

Then, they construct Parseval frame wavelets associated with the MRA structure resulting from ψ_0 by means of the Unitary Extension Principle (UEP) [9, 19, 36]. We briefly describe one aspect of this principle. By *refinable* we mean that there is a 1-periodic function H_0 such that

$$\widehat{\psi_0}(2\gamma) = H_0(\gamma)\,\widehat{\psi_0}(\gamma), \text{a.e. in } \widehat{\mathbb{R}}.$$

The fact that $\widehat{\psi_0}$ is essentially bounded, coupled with the previous equation, imply that H_0 is also essentially bounded. Let H_i, $1 \leq i \leq m$ be essentially bounded 1-periodic functions and define ψ_i, $1 \leq i \leq n$, by $\widehat{\psi_i}(2\gamma) = H_i(\gamma)\,\widehat{\psi_0}(\gamma)$ a.e. in $\widehat{\mathbb{R}}$. Next, define the matrix

$$H(\gamma) = (H_{i,\epsilon}(\gamma))_{i=0,1,...,n;\epsilon=0,1}$$

where $H_{i,\epsilon}(\gamma) = H_i\left(\gamma + \frac{\epsilon}{2}\right)$. If $H^*H = I_{2\times 2}$ a..e on

$$\left\{\gamma \in \widehat{\mathbb{R}} : \widehat{\psi_0}(\gamma+k) \neq 0 \text{ for some } k \in \mathbb{Z}\right\},$$

then $\left\{2^{j/2}\psi_i\left(2^j t - k\right) : j,k \in \mathbb{Z}, i = 1,...,n\right\}$ is a Parseval frame for $L^2(\mathbb{R})$. The wavelet frames produced by this principle are not necessarily semi-orthogonal, thereby generalizing a theorem by Lawton. A more recent generalization of Lawton's theorem can be found in [4].

2. Definitions and preliminaries

2.1. **Definitions.**

- The *Fourier transform* $\widehat{f} : \widehat{\mathbb{R}}^d \longrightarrow \mathbb{C}$ of $f \in L^1\left(\mathbb{R}^d\right)$ is defined by

$$\forall \gamma \in \widehat{\mathbb{R}}^d,\ \widehat{f}(\gamma) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot\gamma} dx.$$

 $\widehat{\mathbb{R}}^d$ is $\mathbb{R}^d$ considered as the spectral domain of the Fourier transform, and $x \cdot \gamma$ denotes the standard inner product on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. The map $f \to \widehat{f}$ restricted to $L^1\left(\mathbb{R}^d\right) \cap L^2\left(\mathbb{R}^d\right)$ extends to a unitary map on $L^2\left(\mathbb{R}^d\right)$.
- A *frame* for a separable Hilbert space H is a sequence $\{f_i\}_{i\in\mathbb{I}} \subseteq H$, where $\mathbb{I}$ is a countable index set, for which there are constants $A, B > 0$ such that

$$\forall f \in H,\ A||f||^2 \leq \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \leq B||f||^2.$$

- A *Parseval frame* for a separable Hilbert space H is a frame in which $A = B = 1$.

- A *Riesz sequence* or *exact frame* is a sequence $\{f_i\}_{i\in\mathbb{I}} \subseteq H$, for which there are constants $A, B > 0$ such that

$$A\sum_n |c_n|^2 \leq \left\|\sum_n c_n f_n\right\|^2 \leq B\sum_n |c_n|^2 .$$

 for all sequences $\{c_n\}$ with finite number of nonzero entries.
- We define the map $\mathcal{X}$ in the following way: for $f \in L^2\left(\mathbb{R}^d\right)$,

$$\mathcal{X}(f)(\gamma) = \left\{\widehat{f}(\gamma+k)\right\}_{k\in\mathbb{Z}^d}.$$

 Then $\mathcal{X}(f)(\gamma) \in l^2\left(\mathbb{Z}^d\right)$ for almost every γ in $\widehat{\mathbb{R}}^d$.
- The periodization of $|\widehat{\varphi}|^2$ is defined as $\Phi(\gamma) = \|\mathcal{X}(\varphi)(\gamma)\|^2_{l^2(\mathbb{Z}^d)}$.
- τ_y is the translation operator defined by $\tau_y f(x) = f(x-y)$.
- $A'\left(\mathbb{Z}^d\right)$ is the linear space of all sequences of Fourier coefficients of bounded periodic functions. It is called the space of *pseudomeaures* on $\mathbb{Z}^d$.
- $\mathcal{L}^\infty$ is the linear space of measurable functions $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ for which

$$\exists B = B(f) \text{ such that } \sum_{n\in\mathbb{Z}^d} |\tau_n f| \leq B \text{ a.e.}$$

Remark 1. *It is clear that if* $\varphi \in L^2\left(\mathbb{R}^d\right)$, *then* $\Phi \in L^1\left(\mathbb{T}^d\right)$; *and* $\|\Phi\|_{L^1(\mathbb{T}^d)} = \|\varphi\|^2_{L^2(\mathbb{R}^d)}$ *by the Parseval-Plancherel theorem.*

Definition 1. *A* frame multiresolution analysis *(FMRA)* $(V_j, \varphi)_{j\in\mathbb{Z}}$ *of* $L^2(\mathbb{R}^d)$ *is an increasing sequence of closed linear subspaces* $V_j \subset L^2(\mathbb{R}^d)$ *and an element* $\varphi \in V_0$ *for which the following hold:*

(1) $\overline{\cup_j V_j} = L^2(\mathbb{R}^d)$ *and* $\cap V_j = \{0\}$,
(2) $f \in V_j \Longleftrightarrow Df \in V_{j+1}$, *where* $Df(x) = 2^{d/2} f(2x)$,
(3) $\forall k \in \mathbb{Z}^d, f \in V_0 \Longleftrightarrow \tau_k f \in V_0$,
(4) $\{\tau_k\varphi : k \in \mathbb{Z}^d\}$ *is a frame for* V_0.

2.2. **Preliminaries.** For $\varphi \in L^2(\mathbb{R}^d)$, let $V_0 = \overline{\text{span}}\{\tau_n\varphi : n\in\mathbb{Z}^d\}$ be the closed linear span of the sequence$\{\tau_n\varphi\}_{n\in\mathbb{Z}^d}$. Then it is elementary to prove that $\{\tau_n\varphi\}$ is an exact frame (or a Riesz basis) for its closed linear span if and only if there exist constants A, B, with $0 < A \leq B < \infty$ for which

$$A \leq \Phi \leq B \text{ a.e.}$$

A similar result is true for frames of translates [6, 9, 16, 17, 36]. In order to state this latter result we use the"pullback" notation $[f > 0]$ to designate the set of points x in the domain of f for which f is positive. Then $\{\tau_n\varphi\}_{n\in\mathbb{Z}^d}$ is a frame for its closed linear span if and only if there exist constants A, B, with $0 < A \leq B < \infty$ for which

$$A \leq \Phi \leq B \text{ a.e. on } [\Phi > 0].$$

This result can be generalized in the case of several $\varphi_i s$ in terms of the *Gramian matrix*

$$G_X(\gamma) = \left(\sum_{n\in\mathbb{Z}^d} \widehat{\varphi_k}(\gamma-n)\,\overline{\widehat{\varphi_j}(\gamma-n)}\right)_{1\leq k,j\leq N} = \left(\langle \mathcal{X}(\varphi_k)(\gamma), \mathcal{X}(\varphi_j)(\gamma)\rangle\right)_{1\leq k,j\leq N},$$

where $X = \{\varphi_i\}_{i=1}^N$. Let $m(\gamma), m^+(\gamma)$, and $M(\gamma)$ be the smallest, smallest positive, and largest eigenvalues of $G_X(\gamma)$, respectively. Then $\{\tau_n\varphi_i : n \in \mathbb{Z}^d, 1 \le i \le N\}$ is a frame for its closed linear span if and only if there exist constants A, B, with $0 < A \le B < \infty$ for which

$$A \le m^+(\gamma) \le M(\gamma) \le B \text{ a.e.}$$

in a particular set called the *spectrum* of $\overline{\text{span}}_{k\in\mathbb{Z}^d}\ \{\tau_k\varphi : \varphi \in X\}$ [12, 13, 14, 15]. The definition of spectrum will be given in section 3. In the case that the translations of elements of X form a Riesz basis, $m^+(\gamma)$ can be replaced by $m(\gamma)$, and the inequality holds a.e.

3. Shift invariant subspaces

We shall use the shift-invariant approach in Section 5 to prove Theorem 9, and we assemble some required results in this section. For more details about shift-invariant subspace theory we refer to [12, 13, 14, 15, 23].

If $W \subset L^2(\mathbb{R}^d)$ and $\gamma \in \widehat{\mathbb{R}}^d$, we set

$$\mathcal{X}(W)(\gamma) = \{(\mathcal{X}(f))(\gamma) : f \in W\},$$

and hence $\mathcal{X}(W)(\gamma) \subset l^2(\mathbb{Z}^d)$ for almost all γ. If $W \subset L^2(\mathbb{R}^d)$ is a linear subspace of $L^2(\mathbb{R}^d)$, then, by the linearlity of $\mathcal{X}$, $\mathcal{X}(W)(\gamma)$ is a linear subspace of $l^2(\mathbb{Z}^d)$. $Sp_{\mathcal{X},\gamma}(W)$ is defined to be

$$Sp_{\mathcal{X},\gamma}(W) = \overline{\text{span}}\,\{(\mathcal{X}(f))(\gamma) : f \in W\}.$$

For $W \subset L^2(\mathbb{R}^d)$, we define $S(W)$ as

$$S(W) = \overline{\text{span}}_{k\in\mathbb{Z}^d}\,\{\tau_k f : f \in W\},$$

the *shift-invariant space generated by* W. If W is a finite set, we say that $S = S(W)$ is a *finitely generated shift-invariant space* (FSI). The *length* of a shift-invariant subspace S is defined to be $\text{len } S = \min \text{card}\{W : S = S(W)\}$. For S a shift-invariant subspace of $L^2(\mathbb{R}^d)$, the *spectrum of* S, $\sigma(S)$, is defined by

$$\sigma(S) = \{\gamma \in \mathbb{T}^d : \dim Sp_{\mathcal{X},\gamma}(W) > 0\},$$

where dim indicates dimension. Note that $\sigma(V_0) = [\Phi > 0] = \{\gamma \in \mathbb{T}^d : \Phi(\gamma) > 0\}$ for the case of an FMRA. Let H be a Hilbert space and let F be a linear subspace of H. Then $F^\perp$, the *orthogonal complement of* F *on* H, is defined as

$$F^\perp = \{x \in H : \forall f \in F,\ \langle x, f\rangle = 0\}.$$

The continuity of the inner product implies that $F^\perp$ is a closed linear subspace of H. We now state some results which we shall need in section 5.

Theorem 1. *Let S be a finitely generated shift-invariant space and let T be a shift-invariant subspace of S. Then $T^\perp$ is also shift-invariant and, for almost every $\gamma \in \widehat{\mathbb{R}}^d$,*

$$\mathcal{X}(S)(\gamma) = \mathcal{X}(T)(\gamma) \bigoplus \mathcal{X}(T^\perp)(\gamma).$$

Theorem 2. *Given any FSI S, there is a finite subset $W \subset L^2(\mathbb{R}^d)$, for which the multi-integer translates of W are a frame for S.*

Theorem 3. *For a shift-invariant subspace $S \subset L^2\left(\mathbb{R}^d\right)$, we have*

$$\text{len } S = \textit{ess-sup}\left\{\dim \mathcal{X}\left(S\right)\left(\gamma\right), \gamma \in \mathbb{T}^d\right\}.$$

The map $D_S\left(\gamma\right) = \dim \mathcal{X}\left(S\right)\left(\gamma\right)$ is the dimension function *of the subspace S.*

These theorems together with their proofs can be found in [12], [36], and [13], respectively.

4. FMRA Frames

The following results are well known for the case $d = 1$, see [8, 9]. The equations in these results are the key for the construction of FMRA frames.

Theorem 4. *Let (V_j, φ) be an FMRA, let $\omega = \{\psi_1, ..., \psi_m\} \subset W_0$, the orthogonal complement of V_0 in V_1, and set $\psi_0 = \varphi$.*

(1) *If $\cup_{k\in\mathbb{Z}^d}\tau_k\omega = \{\tau_k\psi_p : 1 \le p \le m;\ k \in \mathbb{Z}^d\}$ is a frame for W_0, then there exist $g_0, ..., g_m \in l^2\left(\mathbb{Z}^d\right)$ such that*

$$\forall n\in\mathbb{Z}^d,\ \varphi\left(2x-n\right) = \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p\left(2k-n\right)\psi_p\left(x-k\right) \text{ in } L^2(\mathbb{R}^d). \tag{4.1}$$

(2) *If there exist $g_0, ..., g_m \in A'\left(\mathbb{Z}^d\right)$ such that (4.1) is valid, and if $\left\|\mathcal{X}\left(\psi_p\right)\left(\gamma\right)\right\|^2_{l^2(\mathbb{Z}^d)}$ is essentially bounded for each $1 \le p \le m$, i.e., each $\widehat{\psi_p}^2 \in \mathcal{L}^\infty$, then $\cup_{k\in\mathbb{Z}^d}\tau_k\omega$ is a frame for W_0.*

Proof. Any $f \in V_1$ can be written uniquely as $f_0 + k_0$, with $f_0 \in V_0$ and $k_0 \in W_0$. For each $m \in \mathbb{Z}^d$ and for each $u\in\{0,1\}^d$, $\varphi\left(2x-2m-u\right)$ is an element of V_1. Since $\cup_{k\in\mathbb{Z}^d}\tau_k\omega = \{\tau_k\psi_p : 1 \le p \le m;\ k \in \mathbb{Z}^d\}$ generates W_0, and since $\cup_{k\in\mathbb{Z}^d}\tau_k\varphi$ is a frame for V_0, there exists a set $\left\{g_{i,u} \in l^2\left(\mathbb{Z}^d\right) : 0 \le i \le m; u\in\{0,1\}^d\right\}$ such that for each $m \ge 0$ and each $u\in\{0,1\}^d$, we have

$$\varphi\left(2x-2m-u\right) = \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_{p,u}\left(k-m\right)\psi_p\left(x-k\right).$$

Now, define $\{g_p\}_{p=0}^m$ by means of the formula

$$g_p\left(2k+u\right) = g_{p,u}\left(k\right),\ 0 \le p \le m,\ u\in\{0,1\}^d,\ k \in \mathbb{Z}^d.$$

If $n = 2l + u$, $n, l \in \mathbb{Z}^d, u\in\{0,1\}^d$, then

$$\begin{aligned}\varphi\left(2x-n\right) = \varphi\left(2x-2l-u\right) &= \sum_{i=0}^{m}\sum_{k\in\mathbb{Z}^d} g_{p,u}\left(k-l\right)\psi_p\left(x-k\right)\\ &= \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p\left(2k-2l-u\right)\psi_p\left(x-k\right)\\ &= \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p\left(2k-n\right)\psi_p\left(x-k\right) \text{ in } L^2(\mathbb{R}^d).\end{aligned}$$

Thus, the proof of (1) is complete.

Next, assume that the hypotheses of part (2) hold. For each $f \in W_0 \subset V_1$, there is $\{c(n)\}_{n\in\mathbb{Z}^d} \in l^2\left(\mathbb{Z}^d\right)$ such that

$$\begin{aligned} f(x) &= \sum_{n\in\mathbb{Z}^d} c(n)\,\varphi(2x-n) \\ &= \sum_{u\in\{0,1\}^d}\sum_{k\in\mathbb{Z}^d} c(2k+u)\,\varphi(2x-2k-u). \end{aligned}$$

The previous equation is equivalent to

$$\begin{aligned} \widehat{f}(\gamma) &= \sum_{k\in\mathbb{Z}^d} 2^{-d} \sum_{u\in\{0,1\}^d} c(2k+u)\,\widehat{\varphi}\left(\tfrac{\gamma}{2}\right) e^{-2\pi ik\cdot\gamma} e^{-\pi iu\cdot\gamma} \\ &= \sum_{u\in\{0,1\}^d} 2^{-d}\left[\sum_{k\in\mathbb{Z}^d} c(2k+u)\,e^{-2\pi ik\cdot\gamma}\right] e^{-\pi iu\cdot\gamma}\widehat{\varphi}\left(\tfrac{\gamma}{2}\right) \qquad (4.2) \\ &= \sum_{u\in\{0,1\}^d} e^{-\pi iu\cdot\gamma}\widehat{\varphi}\left(\tfrac{\gamma}{2}\right) C_u(\gamma) \text{ in } L^2(\widehat{\mathbb{R}}^d), \end{aligned}$$

where $C_u(\gamma) = 2^{-d}\sum_{k\in\mathbb{Z}^d} c(2k+u)\,e^{-2\pi ik\cdot\gamma} \in L^2(\mathbb{T}^d)$, and where the convergence in $L^2(\widehat{\mathbb{R}}^d)$ is in terms of the partial sums of the C_u by the Parseval-Plancherel theorem. If we take the Fourier transform of (4.1), we obtain for $n = 2l+u$, $u\in\{0,1\}^d$, and $l \in \mathbb{Z}^d$, that

$$\varphi(2x-2l-u) = \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p(2(k-l)-u)\,\psi_p(x-k)$$

if and only if

$$2^{-d}\widehat{\varphi}\left(\tfrac{\gamma}{2}\right) e^{-2\pi il\cdot\gamma} e^{-\pi iu\cdot\gamma} = \sum_{p=0}^{m}\left[\sum_{k\in\mathbb{Z}^d} g_p(2(k-l)-u)\,e^{-2\pi i\mathbf{k}\cdot\gamma}\right]\widehat{\psi_p}(\gamma).$$

Hence,

$$\begin{aligned} \widehat{\varphi}\left(\tfrac{\gamma}{2}\right) e^{-\pi iu\cdot\gamma} &= 2^d\sum_{p=0}^{m}\left[\sum_{k\in\mathbb{Z}^d} g_p(2(k-l)-u)\,e^{-2\pi i(k-l)\cdot\gamma}\right]\widehat{\psi_p}(\gamma) \\ &= \sum_{p=0}^{m} G_{p,u}(\gamma)\,\widehat{\psi_p}(\gamma), \end{aligned}$$

where $G_{p,u}(\gamma) = 2^d\sum_{k\in\mathbb{Z}^d} g_p(2k-u)\,e^{-2\pi ik\cdot\gamma} \in L^2(\mathbb{T}^d)$, for all p, u, and the convergence in $L^2(\widehat{\mathbb{R}}^d)$ is in terms of the partial sums of the $G_{p,u}s$. Substituting into

equation (4.2), we have

$$\begin{aligned}\widehat{f}(\gamma) &= \sum_{u\in\{0,1\}^d}\left(e^{-\pi iu\cdot\gamma}\widehat{\varphi}\left(\tfrac{\gamma}{2}\right)\right)C_u(\gamma)\\ &= \sum_{u\in\{0,1\}^d}\left(\sum_{j=0}^{m}G_{p,u}(\gamma)\,\widehat{\psi_j}(\gamma)\right)C_u(\gamma)\\ &= \sum_{p=0}^{m}\left(\sum_{u\in\{0,1\}^d}G_{p,u}(\gamma)\,C_u(\gamma)\right)\widehat{\psi_j}(\gamma)\\ &= \sum_{p=0}^{m}E_p(\gamma)\,\widehat{\psi_p}(\gamma),\end{aligned}$$

where $E_p(\gamma)=\sum_{u\in\{0,1\}^d}G_{p,u}(\gamma)\,C_u(\gamma)$. Using the hypothesis that $g_p\in A'\left(\mathbb{Z}^d\right), 0\leq p\leq m$, we see that $E_p\in L^2(\mathbb{T}^d)$. Now, for $N\in\mathbb{N}$, we define

$$\widehat{f}_N(\gamma)=\sum_{p=0}^{m}\left[\sum_{|n|\leq N}E_p^{\vee}(n)\,e^{-2\pi in\cdot\gamma}\right]\widehat{\psi_p}(\gamma)=\sum_{p=0}^{m}E_p^N(\gamma)\,\widehat{\psi_p}(\gamma),$$

where $E_p^N(\gamma)=\sum_{|n|\leq N}E_p^{\vee}(n)\,e^{-2\pi in\cdot\gamma}$ (the N-th symmetric partial sum of $E_p(\gamma)$), $n=(n_1,...,n_d)\in\mathbb{Z}^d$, and $|n|=\sum_{i=1}^{d}|n_i|$. Clearly, $\widehat{f}_N\in L^2(\widehat{\mathbb{R}}^d)$ and

$$\lim_{N\to\infty}||\widehat{f_N}-\widehat{f}||_{L^2(\widehat{\mathbb{R}}^d)}=0,$$

since each $\widehat{\psi_p}^2\in\mathcal{L}^\infty$ and since

$$\begin{aligned}||\widehat{f_N}-\widehat{f}||_{L^2(\widehat{\mathbb{R}}^d)} &\leq \sum_{p=0}^{m}||\widehat{\psi_p}\left(E_p-E_p^N\right)||_{L^2(\widehat{\mathbb{R}}^d)}\\ &= \sum_{p=0}^{m}\sum_{n\in\mathbb{Z}^d}\left(\int_{[0,1]^d}\tau_n|\widehat{\psi_p}|^2\left|E_p-E_p^N\right|^2\right)^{\frac{1}{2}}\\ &= \sum_{p=0}^{m}||\sqrt{\Phi_p}\left(E_p-E_p^N\right)||_{L^2(\mathbb{T}^d)}\\ &\leq \sum_{p=0}^{m}||\sqrt{\Phi_p}||_{L^\infty(\mathbb{T}^d)}||\left(E_p-E_p^N\right)||_{L^2(\mathbb{T}^d)},\end{aligned}$$

where $\Phi_p=P\left(|\widehat{\psi_p}|^2\right), 0\leq p\leq m$. Hence, the Parseval-Plancherel theorem implies that

$$f=\sum_{p=0}^{m}\sum_{n\in\mathbb{Z}^d}E_p^{\vee}(n)\,\tau_n\psi_p \text{ in } L^2(\mathbb{R}^d).$$

Because $f\in W_0$, $\sum_{n\in\mathbb{Z}^d}E_0^{\vee}(n)\,\tau_n\psi_0=\sum_{n\in\mathbb{Z}^d}E_0^{\vee}(n)\,\tau_n\varphi=0$. Hence,

$$f=\sum_{p=1}^{m}\sum_{n\in\mathbb{Z}^d}E_p^{\vee}(n)\,\tau_n\psi_p \text{ in } L^2(\mathbb{R}^d),$$

where $\left\{E_p^{\vee}(n)\right\}_{n\in\mathbb{Z}^d} \in l^2\left(\mathbb{Z}^d\right)$, for $p \in \{1,...,m\}$. So far we have proved that $\cup_{k\in\mathbb{Z}^d}\tau_k\omega$ generates W_0. This mean that the linear operator $T^*_{W_0} : l^2\left(\mathbb{Z}^d\right) \to W_0$, $T^*_\omega\left(\{c_k\}\right) = \sum_{p=1}^m \sum_k c_k\tau_k\psi_p$, is a surjection onto W_0. Its adjoint, T_{W_0}, is bounded since $\widehat{\psi_p}^2 \in \mathcal{L}^\infty$, $1 \le p \le m$. This implies that $T^*_{W_0}$ is also bounded. The open mapping theorem guarantees that $T^*_{W_0}$ is also bounded below on $\mathcal{N}\left(T^*_{W_0}\right)^\perp$, since the restriction of this map to $\mathcal{N}\left(T^*_{W_0}\right)^\perp$ is an invertible operator. It follows that $\cup_{k\in\mathbb{Z}^d}\tau_k\omega$ is a frame for W_0 (see Proposition 3.4 in [9], and Theorem 5.5.1 in [17]). □

The following result expresses the orthogonality relations between V_0 and W_0 in terms of filters.

Theorem 5. *Let $H_0 \in L^\infty\left(\mathbb{T}^d\right)$, and let (V_j, φ) be an FMRA, where H_0 and φ satisfy*

$$\widehat{\varphi}(\gamma) = H_0\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right) \quad a.e.\ in\ \ L^2(\widehat{\mathbb{R}}^d). \tag{4.3}$$

Define W_0 as the orthogonal complement of V_0 in V_1. Further, for $\{h_1[n]\} \in l^2\left(\mathbb{Z}^d\right)$, let $\widehat{h_1} = H_1 \in L^2\left(\mathbb{T}^d\right)$, and let $\psi \in V_1$ be defined as

$$\widehat{\psi}(\gamma) = H_1\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right) \quad a.e.\ in\ \ L^2(\widehat{\mathbb{R}}^d). \tag{4.4}$$

Then $\psi \in W_0$ if and only if

$$\sum_{u\in\{0,1\}^d} \tau_{\frac{u}{2}}\left(H_1\overline{H_0}\Phi\right) = 0 \quad a.e.\ in\ L^2(\mathbb{T}^d). \tag{4.5}$$

Proof. Set $e_k(\gamma) = e^{-2\pi ik\cdot\gamma}$. Then $\psi \in V_0^\perp$ *(in V_1)*

$$\begin{aligned}
&\iff \forall k \in \mathbb{Z}^d, \left\langle\widehat{\psi}, e_k\widehat{\varphi}\right\rangle = \langle\psi, \tau_k\varphi\rangle = 0\\
&\iff \forall k \in \mathbb{Z}^d, \left\langle H_1\left(\tfrac{\cdot}{2}\right)\widehat{\varphi}\left(\tfrac{\cdot}{2}\right), e_kH_0\left(\tfrac{\cdot}{2}\right)\widehat{\varphi}\left(\tfrac{\cdot}{2}\right)\right\rangle = 0\\
&\iff \forall k \in \mathbb{Z}^d, \langle H_1\widehat{\varphi}, e_{2k}H_0\widehat{\varphi}\rangle = 0\\
&\iff \forall k \in \mathbb{Z}^d, \int_{\mathbb{R}^d} H_1\overline{H_0}|\widehat{\varphi}|^2e_{2k} = 0\\
&\iff \forall k \in \mathbb{Z}^d, \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} H_1\overline{H_0}\Phi e_{2k} = 0\\
&\iff \forall k \in \mathbb{Z}^d, \sum_{u\in\{0,1\}^d}\int_{\left[0,\frac{1}{2}\right]^d} \tau_{\frac{u}{2}}\left(H_1\overline{H_0}\Phi\right)e_{2k} = 0\\
&\iff \int_{\left[0,\frac{1}{2}\right]^d}\sum_{u\in\{0,1\}^d} \tau_{\frac{u}{2}}\left(H_1\overline{H_0}\Phi\right)e_{2k} = 0.
\end{aligned}$$

The result follows by the L^1- uniqueness theorem for Fourier series. □

Remark 2. *After periodizing the modulus squared of (4.3), equation (4.3) is equivalent to the following equation:*

$$\Phi = \sum_{u\in\{0,1\}^d} \tau_{-\frac{u}{2}}\left[\left(\left|H_0\right|^2\Phi\right)\left(\frac{\cdot}{2}\right)\right]. \tag{4.6}$$

This equation will be needed for a remark after Theorem 9 of Section 5.

Equation (4.7) gives rise to the fast wavelet algorithm associated with an FMRA to which we referred in the Introduction. This equation gives an exact reconstruction formula provided that the frequency content of the decomposed multidimensional signal has support contained in $[\Phi > 0]$. As mentioned in the Introduction, the filters H_p are called analysis filters and the filters G_p are called synthesis filters.

Theorem 6. *Let (V_j, φ) be an FMRA, let $H_0 \in L^\infty(\mathbb{T}^d)$ satisfy $\widehat{\varphi}(\gamma) = H_0\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right)$ in $L^2(\widehat{\mathbb{R}}^d)$, and let $\omega = \{\psi_1, ..., \psi_m\} \subset W_0$ and $H_p \in L^2(\mathbb{T}^d)$, $1 \le p \le m$, satisfy $\widehat{\psi_p}(\gamma) = H_p\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right)$ in $L^2(\widehat{\mathbb{R}}^d)$. Suppose that $\widehat{g_p} = G_p \in L^\infty(\mathbb{T}^d)$, $0 \le p \le m$. Then (4.1) holds if and only if*

$$\forall u \in \{0,1\}^d,\ \sum_{p=0}^{m} \tau_{-\frac{1}{2}u}(H_p\Phi)G_p = \Phi\delta(0,u). \tag{4.7}$$

Here, $\delta(0,u)$ is the Kronecker delta.

Proof. Equation (4.1) is equivalent to the following equation:

$$\begin{aligned}\forall\ n \in \mathbb{Z}^d,\ \widehat{\varphi}(\gamma) &= 2^d \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p(2k-n)\, e^{-2\pi i(2k-n)\cdot\gamma}\widehat{\psi_p}(2\gamma)\\ &= 2^d \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p(2k-n)\, e^{-2\pi i(2k-n)\cdot\gamma} H_p(\gamma)\,\widehat{\varphi}(\gamma). \qquad (4.8)\end{aligned}$$

Adding these equations over all $v \in \{0,1\}^d$ we obtain

$$\begin{aligned}2^d\widehat{\varphi}(\gamma) &= 2^d \sum_{p=0}^{m}\sum_{v\in\{0,1\}^d}\sum_{k\in\mathbb{Z}^d} g_p(2k-v)\, e^{-2\pi i(2k-v)\cdot\gamma} H_p(\gamma)\,\widehat{\varphi}(\gamma)\\ &= 2^d \sum_{p=0}^{m}\sum_{q\in\mathbb{Z}^d} g_p(q)\, e^{-2\pi iq\cdot\gamma} H_p(\gamma)\,\widehat{\varphi}(\gamma)\\ &= 2^d \sum_{p=0}^{m} G_p(\gamma)\, H_p(\gamma)\widehat{\varphi}(\gamma).\end{aligned}$$

Here, $G_p(\gamma) = \sum_{q\in\mathbb{Z}^d} g_p(q)\, e^{-2\pi iq\cdot\gamma}$, $0 \le p \le m$. The second equality follows from the fact that any $q \in \mathbb{Z}^d$ can be written as $2k - v$, where $k \in \mathbb{Z}^d$ and $v \in \{0,1\}^d$ are uniquely determined. Now, let $v \in \{0,1\}^d$ and $u \in \{0,1\}^d \setminus \{0\}$ be fixed. Multiplying (4.8) by $e^{-\pi iv\cdot u}$, we have

$$\begin{aligned}\widehat{\varphi}(\gamma)e^{-\pi iv\cdot u} &= 2^d \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p(2k-v)\, e^{-2\pi i(2k-v)\cdot\gamma} e^{-\pi iv\cdot u} H_p(\gamma)\,\widehat{\varphi}(\gamma)\\ &= 2^d \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p(2k-v)\, e^{-2\pi i(2k-v)\cdot\left(\gamma-\frac{u}{2}\right)} H_p(\gamma)\,\widehat{\varphi}(\gamma).\end{aligned}$$

On the other hand, $\sum_{v\in\{0,1\}^d} e^{-\pi iv\cdot u} = 0$ for a fixed $u \in \{0,1\}^d \setminus \{0\}$ since $e^{-\pi iv\cdot u} = (-1)^{v\cdot u}$. In fact, $\sum_{v\in\{0,1\}^d} e^{-\pi iv\cdot u} = \sum_{v\in\{0,1\}^d} (-1)^{v\cdot u}$; and so, for a fixed $u \in \{0,1\}^d \setminus \{0\}$, half of the $\{v\cdot u\}_{v\in\{0,1\}^d}$ are even and half are odd, and so

the original sum of exponentials is zero. Hence,

$$\begin{aligned}
0 &= (\sum_{v\in\{0,1\}^d} e^{-\pi i v\cdot u})\widehat{\varphi}(\gamma) \\
&= 2^d \sum_{p=0}^{m} \sum_{v\in\{0,1\}^d} \sum_{k\in\mathbb{Z}^d} g_p\,(2k-v)\, e^{-2\pi i(2k-v)\cdot(\gamma-\frac{u}{2})} H_p\,(\gamma)\,\widehat{\varphi}(\gamma) \\
&= 2^d \sum_{p=0}^{m} \sum_{q\in\mathbb{Z}^d} g_p\,(q)\, e^{-2\pi i q\cdot(\gamma-\frac{u}{2})} H_p\,(\gamma)\,\widehat{\varphi}(\gamma) \\
&= 2^d \sum_{p=0}^{m} G_p(\gamma-\frac{u}{2}) H_p(\gamma)\widehat{\varphi}(\gamma) \\
&= 2^d \sum_{p=0}^{m} G_p\,(\xi)\, H_p(\xi+\frac{u}{2})\widehat{\varphi}(\xi+\frac{u}{2}).
\end{aligned}$$

Summarizing the previous calculations, we have verified that

$$\widehat{\phi}(\xi) = \sum_{p=0}^{m} G_p\,(\xi)\, H_p(\xi)\widehat{\varphi}(\xi),\ \text{a.e. } \xi\in\widehat{\mathbb{R}}^d \tag{4.9}$$

and

$$0 = \sum_{p=0}^{m} G_p\,(\xi)\, H_p(\xi+\frac{u}{2})\widehat{\varphi}(\xi+\frac{u}{2}),\ \text{a.e. } \xi\in\widehat{\mathbb{R}}^d \tag{4.10}$$

are equivalent to (4.1).

To prove the theorem, assume (4.1), i.e., assume (4.9) and (4.10). Multiplying (4.9) and (4.10) by $\overline{\widehat{\varphi}(\xi)}$ and $\overline{\widehat{\varphi}(\xi+\frac{u}{2})}$, respectively, we obtain

$$|\widehat{\phi}(\xi)|^2 = \sum_{p=0}^{m} G_p\,(\xi)\, H_{pj}(\xi)|\widehat{\varphi}(\xi)|^2 \text{ a.e. } \xi\in\widehat{\mathbb{R}}^d$$

and

$$0 = \sum_{p=0}^{m} G_p\,(\xi)\, H_p(\xi+\frac{u}{2})|\widehat{\varphi}(\xi+\frac{u}{2})|^2,\ \text{a.e. } \xi\in\widehat{\mathbb{R}}^d.$$

By the periodicity of the G_ps and the H_ps, the change of variable $\xi\to\xi+k$ produces

$$|\widehat{\varphi}(\xi+k)|^2 = \sum_{p=0}^{m} G_p\,(\xi)\, H_p(\xi)|\widehat{\varphi}(\xi+k)|^2,\ \text{a.e. } \xi\in\widehat{\mathbb{R}}^d$$

and

$$0 = \sum_{p=0}^{m} G_p\,(\xi)\, H_p(\xi+\frac{u}{2})|\widehat{\varphi}(\xi+k+\frac{u}{2})|^2,\ \text{a.e. } \xi\in\widehat{\mathbb{R}}^d.$$

Summing over all k in $\mathbb{Z}^d$, we have

$$\Phi(\xi) = \sum_{p=0}^{m} G_p\,(\xi)\, H_p(\xi)\Phi(\xi),\ \text{a.e. } \xi\in\widehat{\mathbb{R}}^d$$

and

$$0=\sum_{p=0}^{m} G_p\left(\xi\right) H_p(\xi+\frac{u}{2})\Phi(\xi+\frac{u}{2}),\ \text{a.e. } \xi\in\widehat{\mathbb{R}}^d,$$

i.e.,

$$\sum_{p=0}^{m}\tau_{-\frac{1}{2}u}(H_p\Phi)G_p=\Phi\delta\left(0,u\right),\ \text{a.e. on } \mathbb{T}^d.$$

This proves the"only if" part.

Conversely, if equation (4.7) has a solution $G_p\in L^\infty(\mathbb{T}^d)$, $0\le p\le m$, then

$$1=\sum_{p=0}^{m} G_p\left(\xi\right) H_p(\xi),\ \text{a.e. } \xi\in[\Phi>0]$$

and

$$0=\sum_{p=0}^{m} G_p\left(\xi\right) H_p(\xi+\frac{u}{2}),\ \text{a.e. } \xi\in\left[\tau_{-\frac{u}{2}}\Phi>0\right].$$

Clearly, $\widehat{\varphi}(\xi)\neq 0$ implies that $\Phi(\xi)\neq 0$; and therefore $[\widehat{\varphi}\neq 0]\subseteq[\Phi>0]$ and $\left[\tau_{-\frac{u}{2}}\varphi\neq 0\right]\subseteq\left[\tau_{-\frac{u}{2}}\Phi>0\right]$. This implies that the previous two equations hold a.e. on $[\Phi>0]$ and on $\left[\tau_{-\frac{u}{2}}\Phi>0\right]$, respectively. This yields (4.9) and (4.10), and, hence, the proof is complete. □

Combining Theorems 4 and 6 we obtain the following result.

Theorem 7. *Let (V_j,φ) be an FMRA, let $\omega=\{\psi_1,...,\psi_m\}\subset W_0$, and let H_p, $0\le p\le m$, be as above. Assume that $\widehat{\psi_p}^2\in\mathcal{L}^\infty$, $1\le p\le m$. If there are $\widehat{g}_p=G_p\in L^\infty(\mathbb{T}^d)$, $0\le p\le m$, such that (4.7) holds, then $\cup_{k\in\mathbb{Z}^d}\tau_k\omega$ is a frame for W_0.*

Remark 3. *The result stated in Theorem 6 can be formulated in terms of matrices: order the set $\{0,1\}^d$ by any order you may wish with the only requirement that*

$$u_0=(0,0,...,0),$$

*(the lexicographical order is one possibility) then, form the **modulation** matrix $\mathbf{H}$ defined in $\mathbb{T}^d$ with entries in $L^\infty(\mathbb{T}^d)$ by*

$$\mathbf{H}\left(\gamma\right)=\begin{pmatrix} H_0\left(\gamma\right) & ... & H_m\left(\gamma\right)\\ H_0\left(\gamma+\frac{u_1}{2}\right) & ... & H_m\left(\gamma+\frac{u_1}{2}\right)\\ : & : & :\\ H_0\left(\gamma+\frac{u_{2^d-1}}{2}\right) & ... & H_m\left(\gamma+\frac{u_{2^d-1}}{2}\right)\end{pmatrix}\quad a.e.$$

Now, we define a new matrix $\mathbf{\Phi}_{2^d\times 2^d}\left(\gamma\right)$ as follows:

$$\mathbf{\Phi}_{2^d\times 2^d}\left(\gamma\right)=\begin{pmatrix}\Phi\left(\gamma\right) & 0 & ... & 0\\ 0 & \Phi\left(\gamma+\frac{u_1}{2}\right) & ... & 0\\ : & : & : & :\\ 0 & 0 & ... & \Phi\left(\gamma+\frac{u_{2^d-1}}{2}\right)\end{pmatrix}\quad a.e.$$

Set $\mathbf{G}$ *for the modulation matrix for the filters* $\overline{G_p} \in L^\infty(\mathbb{T}^d)$, $0 \le p \le m$

$$\mathbf{G}(\gamma) = \begin{pmatrix} \overline{G_0(\gamma)} & \dots & \overline{G_m(\gamma)} \\ \overline{G_0\left(\gamma + \frac{u_1}{2}\right)} & \dots & \overline{G_m\left(\gamma + \frac{u_1}{2}\right)} \\ \vdots & \vdots & \vdots \\ \overline{G_0\left(\gamma + \frac{u_{2^d-1}}{2}\right)} & \dots & \overline{G_m\left(\gamma + \frac{u_{2^d-1}}{2}\right)} \end{pmatrix} \quad a.e.$$

Making the change of variable $\gamma \longrightarrow \gamma + \frac{u_i}{2}$, *equation (4.7) can be rewritten as*

$$\sum_{p=0}^{m} H_p\left(\gamma + \frac{u_i}{2}\right) \Phi\left(\gamma + \frac{u_i}{2}\right) G_p\left(\gamma + \frac{u_j}{2}\right) = \Phi\left(\gamma + \frac{u_j}{2}\right) \delta_{ij} \tag{4.11}$$

for almost every $\gamma \in \mathbb{T}^d$. *Hence, we obtain,*

Theorem 8. *Let* (V_j, φ) *be an FMRA, let* $H_0 \in L^\infty(\mathbb{T}^d)$ *satisfy* $\widehat{\varphi}(\gamma) = H_0\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right)$ *in* $L^2(\widehat{\mathbb{R}}^d)$, *let* $\omega = \{\psi_1, ..., \psi_m\} \subset W_0$ *and* $H_p \in L^2(\mathbb{T}^d)$, $1 \le p \le m$, *satisfy* $\widehat{\psi_p}(\gamma) = H_p\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right)$ *in* $L^2(\widehat{\mathbb{R}}^d)$. *Then,* $\{\tau_k \psi_p : 1 \le p \le m;\ k \in \mathbb{Z}^d\}$ *is a frame for* W_0 *if and only if there exist* $\widehat{g_p} = G_p \in L^\infty(\mathbb{T}^d)$, $0 \le p \le m$, *such that*

$$\mathbf{\Phi}_{2^d \times 2^d}(\gamma)\, \mathbf{H}(\gamma)\, \mathbf{G}(\gamma)^* = \mathbf{\Phi}_{2^d \times 2^d}(\gamma). \tag{4.12}$$

for almost every $\gamma \in \mathbb{T}^d$. *Here,* $\mathbf{\Phi}_{2^d \times 2^d}(\gamma)$, $\mathbf{H}(\gamma)$ *and* $\mathbf{G}(\gamma)$ *are as in the previous remark.*

Remark 4. *Equation (4.12) implies that for almost every* γ *in* $\mathbb{T}^d$, *the columns of* $\mathbf{H}(\gamma)$ *form a frame for the space*

$$\mathcal{M}(\gamma) = (\Phi(\gamma)\,\mathbb{C}) \bigoplus \left(\Phi\left(\gamma + \frac{u_1}{2}\right)\mathbb{C}\right) \bigoplus \dots \bigoplus \left(\Phi\left(\gamma + \frac{u_{2^d-1}}{2}\right)\mathbb{C}\right) = \mathrm{Range}\left(\mathbf{\Phi}_{2^d \times 2^d}(\gamma)\right),$$

since (4.12) implies that the rank $\mathbf{H}(\gamma) \ge \dim \mathcal{M}(\gamma)$, *for almost every* γ *in* $\mathbb{T}^d$. *Take, for instance, an element* $f(\gamma) \in \mathcal{M}(\gamma)$. *Then,* $f(\gamma)$ *can be written as*

$$\begin{aligned} \mathbf{f}(\gamma) &= \left(\Phi(\gamma)\, c_0(\gamma), \Phi\left(\gamma + \frac{u_1}{2}\right) c_1(\gamma), ..., \Phi\left(\gamma + \frac{u_{2^d-1}}{2}\right) c_{2^d-1}(\gamma)\right) \\ &= \mathbf{\Phi}_{2^d \times 2^d}(\gamma)\, \mathbf{C}^T(\gamma), \end{aligned}$$

where

$$\mathbf{C}(\gamma) = (c_0(\gamma), c_1(\gamma), ..., c_{2^d-1}(\gamma)),\ c_j \in L^2(\mathbb{T}^d),\ 0 \le j \le 2^d - 1.$$

Hence,

$$\mathbf{H}(\gamma)\, \mathbf{d}^T(\gamma) = \mathbf{f}^T(\gamma),$$

where

$$\mathbf{d}^T(\gamma) = \mathbf{G}(\gamma)^*\, \mathbf{C}^T(\gamma).$$

If $[\mathbf{A}]_k$ *represents the k-th column of the matrix* $\mathbf{A}$, *a direct computation shows that (4.11) implies*

$$\mathbf{f}^T(\gamma) = \sum_{k=0}^{m} \langle \mathbf{f}(\gamma), [\mathbf{H}(\gamma)]_k \rangle\, [\mathbf{G}(\gamma)]_k \text{ , for almost every } \gamma \in \mathbb{T}^d. \tag{4.13}$$

For instance, the pth entry of the RHS of (4.13) is

$$\sum_{k=0}^{m} \langle \mathbf{f}(\gamma), [\mathbf{H}(\gamma)]_k \rangle G_k\left(\gamma + \frac{u_p}{2}\right)$$
$$= \sum_{k=0}^{m} \sum_{j=0}^{2^d-1} \Phi\left(\gamma + \frac{u_j}{2}\right) c_j(\gamma) \overline{H_k\left(\gamma + \frac{u_j}{2}\right) G_k\left(\gamma + \frac{u_p}{2}\right)}$$
$$= \sum_{j=0}^{2^d-1} c_j(\gamma) \overline{\sum_{k=0}^{m} \Phi\left(\gamma + \frac{u_j}{2}\right) H_k\left(\gamma + \frac{u_j}{2}\right) G_k\left(\gamma + \frac{u_p}{2}\right)}$$
$$= \sum_{j=0}^{2^d-1} c_j(\gamma) \overline{\Phi\left(\gamma + \frac{u_j}{2}\right) \delta_{jp}} = \sum_{j=0}^{2^d-1} c_j(\gamma) \Phi\left(\gamma + \frac{u_j}{2}\right) \delta_{jp} = c_p(\gamma) \Phi\left(\gamma + \frac{u_p}{2}\right) = [\mathbf{f}(\gamma)]_p.$$

This justifies the use of the term dual matrix for $\mathbf{G}(\gamma)$. *We remark that* $[\mathbf{G}(\gamma)]_k$, $k = 0, 1, ..., m$ *do not necessarily form the canonical dual of* $[\mathbf{H}(\gamma)]_k$, $k = 0, 1, ..., m$.

Remark 5. *If the columns of* $\mathbf{H}(\gamma)$ *form a Parseval frame for its span for almost every* γ *in* $\mathbb{T}^d$, *then the frame operator* $S(\gamma)$ *for the columns of* $\mathbf{H}(\gamma)$ *is the identity for almost every* γ *in* $\mathbb{T}^d$. *It follows that*

$$\mathbf{G}(\gamma) = \mathbf{H}(\gamma) \text{ for almost every } \gamma \text{ in } \mathbb{T}^d$$

is one choice for $\mathbf{G}(\gamma)$. *Moreover, if the spectrum of* V_0 *is all of* $\mathbb{T}^d$, *then the* $\Phi\left(\gamma + \frac{u_j}{2}\right)$, $0 \leq j \leq 2^d - 1$ *can be canceled on both sides of the equation (4.11) obtaining the* $2^d \times 2^d$ *identity matrix on the right side of (4.12) and (4.12) takes the form of the Unitary Extension Principle.*

Remark 6. *There are some fundamental differences between the UEP and our approach. First, in the UEP the scaling function is required to be refinable and the translates of it form a Bessel sequence. The frame condition in our assumptions is more restrictive. Second, the wavelet generators are assumed to be in* V_1, *and, in fact, the translates of such generators can span a dense subspace of* V_1. *It follows that the wavelet frames obtained by the UEP are not semi-orthogonal. In our case we obtain semi-orthogonal wavelet frames since we required that our wavelet frames belong to* W_0, *the orthogonal complement of* V_0 *in* V_1.

Example 1. *A radial wavelet frame* [10, 21, 35]*: Let* $\widehat{V_0} = PW_{\{\gamma \in \mathbb{R}^d : \|\gamma\| \leq 1/4\}}$. *Then,* φ *defined by*

$$\widehat{\varphi}(\gamma) = \mathbf{1}_{\{\|\gamma\| \leq 1/4\}}(\gamma)$$

is a scaling function. Also,

$$PW_{\{\gamma \in \mathbb{R}^d : \|\gamma\| \leq 1/2\}} = D^* \widehat{V_0} = \widehat{V_1}.$$

Therefore, following the previous remark, $H_0 = G_0 = \mathbf{1}_{\{\|\gamma\| \leq 1/4\}}$, $H_1 = G_1 = \mathbf{1}_{\{\|\gamma\| \leq 1/4\}} - \mathbf{1}_{\{\|\gamma\| \leq 1/8\}}$ *periodically on* $\widehat{\mathbb{R}}^d$, *and equations (4.7) become*

$$H_0 \Phi G_0 + H_1 \Phi G_1 = \Phi$$

and

$$\forall u \in \{0,1\}^d \setminus \{0\} \quad \tau_{u/2}(H_0 \Phi) G_0 + \tau_{u/2}(H_1 \Phi) G_1 = 0.$$

This equations are trivially satisfied because of the following observation:

$$\left| \left\{ \gamma \in \mathbb{R}^d : \|\gamma\| \leq 1/4 \right\} \cap \tau_{u/2}\left(\left\{ \gamma \in \mathbb{R}^d : \|\gamma\| \leq 1/4 \right\} \right) \right| = 0.$$

A generator ψ for W_0 is given by

$$\widehat{\psi}(\gamma) = \mathbf{1}_{\{\|\gamma\|\leq 1/2\}}(\gamma) - \mathbf{1}_{\{\|\gamma\|\leq 1/4\}}(\gamma).$$

Note that

$$PW_{\{\gamma\in\mathbb{R}^d:\|\gamma\|\leq 1/2\}} = D^*\widehat{V_0} = \widehat{V_1},$$

and

$$\widehat{V_1} = \widehat{V_0}\bigoplus\widehat{W_0},$$

so that $\widehat{W_0} = \widehat{V_0}^{\perp}\cap\widehat{V_1} = PW_{\{\gamma\in\mathbb{R}^d:\|\gamma\|\leq 1/2\}\backslash\{\gamma\in\mathbb{R}^d:\|\gamma\|\leq 1/4\}}$, which is consistent with the above solution. This type of symmetric (radial) nonseparable frame multiresolution analysis has been very useful to detect texture in medical image processing, c.f. [35]*. The filters G_0 and G_1 can be smooth to avoid ringing in the reconstruction proccess.*

Example 2. *More generally, Let A be a subset of $\left[-\frac{1}{2},\frac{1}{2}\right)^d$ that tiles $\mathbb{R}^d$ by dyadic dilation, and define Q by*

$$Q = \cup_{j\geq 0} 2^{-j}A.$$

It is well-known that ψ defined by the following equation

$$\widehat{\psi}(\gamma) = \mathbf{1}_A(\gamma) = \mathbf{1}_Q(\gamma) - \mathbf{1}_{\frac{1}{2}Q}(\gamma)$$

is a Parseval frame wavelet (cf., Lemma 14 in [10]*, Theorem 3.8 in* [21] *). The scaling function φ is given by*

$$\widehat{\varphi}(\gamma) = \mathbf{1}_{\frac{1}{2}Q}(\gamma)$$

and the filters H_0 and H_1 are given by

$$H_0(\gamma) = \mathbf{1}_{\frac{1}{4}Q}(\gamma),\ H_1(\gamma) = \mathbf{1}_{\frac{1}{2}Q}(\gamma) - \mathbf{1}_{\frac{1}{4}Q}(\gamma),\ \text{periodically on } \widehat{\mathbb{R}}^d.$$

The dual filters G_0, G_1 can be taken to be equal to H_0 and H_1 respectively . The equation $\widehat{\psi}(\gamma) = \mathbf{1}_Q(\gamma) - \mathbf{1}_{\frac{1}{2}Q}(\gamma)$ implies that

$$\widehat{\varphi}(\gamma) = \sum_{j\in\mathbb{Z}^d}\left|\widehat{\psi}\left(2^j\gamma\right)\right|^2 < \infty,$$

since $Q\backslash\frac{1}{2}Q$ tiles $\widehat{\mathbb{R}}^d$ by dilations. Hence,

$$\widehat{\psi}(\gamma) = \widehat{\varphi}(\gamma/2) - \widehat{\varphi}(\gamma)$$

and φ is a refinable function associated with ψ. Thus, in light of Theorem 3.8 in [21]*, the wavelet function ψ is derived from φ.*

Once we have a frame for W_0, standard methods can be used to construct an FMRA frame for all of $L^2(\widehat{\mathbb{R}}^d)$, e.g., [6, 7, 8, 9, 17, 19, 26, 27, 28, 36].

5. Measure-theoretic criterion for wavelets and a Mallat-Meyer type algorithm

The main idea of FMRAs is to apply the ideas of classical multiresolution analysis to contruct wavelet frames by means of a generalized Mallat-Meyer algorithm. Theorem 9 is a characterization of when such a construction is possible. As the theorem shows, this construction is not always guaranteed, but depends solely on the measure properties of a certain set which is intimately related to the spectrum of V_0. On the other hand, for the one dimensional case, Kim and Lim construct two

wavelets generating $L^2(\mathbb{R})$ independently of the measure of the aforementioned set [27].

In [8], Benedetto and Li applied the theory of FMRAs to the analysis of narrow band signals. Then, in [9], Benedetto and Treiber presented the main results of the theory of FMRAs from a functional analytic perspective. The proof of the main result in [9] gives a recipe for constructing wavelet frames for $L^2(\mathbb{R})$ when a natural measure theoretic criterion is satisfied, cf., Theorem 9. The construction in [9] can be extended to $\mathbb{R}^d$, for $d > 1$, by tensor products. We shall make this construction in section 6. We emphasize, however, that in the case $d > 1$ Theorem 9 does not provide a recipe for constructing wavelet frames when the appropriate measure theoretic criterion is satisfied. However, Theorem 9 does estimate the minimum number of generators in the worst case scenario, that is, when the measure theoretic criterion is not satisfied:

Theorem 9. *Suppose (V_j, φ) is an FMRA of $L^2(\mathbb{R}^d)$, and let $H_0 \in L^\infty(\mathbb{T}^d)$ have the property that $\widehat{\varphi}(2\gamma) = H_0(\gamma)\,\widehat{\varphi}(\gamma)$ a.e. Set*

$$\Gamma = \left\{\gamma \in \mathbb{T}^d : \Phi(2\gamma) = 0,\ \Phi\left(\gamma + \frac{u}{2}\right) > 0,\ u \in \{0,1\}^d\right\}.$$

Then, there is a set of wavelet functions $\omega = \{\psi_1, ..., \psi_m\} \subset W_0$, $m \leq 2^d - 1$, for which the translations of ω are a frame for W_0 if and only if $|\Gamma| = 0$.

Proof. (H. O. Kim, R. Y. Kim, J. K. Lim)

(a) We shall show that len $V_1 \leq 2^d$. We know that $V_0 = \overline{\text{span}}\left\{\tau_k\varphi : k \in \mathbb{Z}^d\right\}$. Now, $V_1 = DV_0$ which implies that $V_1 = \overline{\text{span}}\left\{D\tau_k\varphi : k \in \mathbb{Z}^d\right\}$. The relations $\forall k \in \mathbb{Z}^d$, $D\tau_{2k} = \tau_k D$ give us

$$D\tau_{2k+u}\varphi = D\tau_{2k}\tau_u\varphi = \tau_k D\tau_u\varphi = \tau_k\varphi_u,\ k \in \mathbb{Z}^d,\ u \in \{0,1\}^d.$$

Here we are using the fact that every $n \in \mathbb{Z}^d$ can be written uniquely in the form $2k + u$, with $k \in \mathbb{Z}^d$ and $u \in \{0,1\}^d$. Also, $\varphi_u = D\tau_u\varphi$. Therefore,

$$V_1 = \overline{\text{span}}\left\{\tau_k\varphi_u : k \in \mathbb{Z}^d,\ u \in \{0,1\}^d\right\}.$$

since card $\{0,1\}^d$ is 2^d, len $V_1 \leq 2^d$.

(b) We shall show that len $V_0 \leq 1$. From $\varphi(2\gamma) = H_0(\gamma)\,\varphi(\gamma)$, we obtain

$$\mathcal{X}(V_0)(\gamma) = \text{span}\left\{\left(H_0\left(\frac{1}{2}(\gamma - k)\right)\widehat{\varphi}\left(\frac{1}{2}(\gamma - k)\right)\right)_{k\in\mathbb{Z}^d}\right\}$$

which implies that $\mathcal{X}(V_0)(\gamma)$ is at most one dimensional :

(c) The $-k$ th component of $\mathcal{X}(\varphi_u)(\gamma)$ is given by $2^{-\frac{d}{2}}e^{-2\pi iu\cdot\frac{1}{2}(\gamma-k)}\widehat{\varphi}\left(\frac{1}{2}(\gamma - k)\right)$ so that

$$\mathcal{X}(V_1)(\gamma) = \text{span}\left\{\left(e^{-2\pi iu\cdot\frac{1}{2}(\gamma-k)}\widehat{\varphi}\left(\frac{1}{2}(\gamma - k)\right)\right)_{k\in\mathbb{Z}^d} : u \in \{0,1\}^d\right\}.$$

If we compute,

$$\begin{aligned}(\mathcal{X}(\varphi_u)(\gamma))_{-k} = \widehat{\varphi}_u(\gamma - k) = \widehat{D\tau_u\varphi}(\gamma - k) &= \int 2^{\frac{d}{2}}\varphi(2x - u)\,e^{-2\pi ix\cdot(\gamma-k)}dx \\ &= \int 2^{-\frac{d}{2}}\varphi(y)\,e^{-2\pi i\frac{1}{2}(y+u)\cdot(\gamma-k)}dy = \int 2^{-\frac{d}{2}}\varphi(y)\,e^{-2\pi i\frac{1}{2}u\cdot(\gamma-k)}e^{-2\pi i\frac{1}{2}y\cdot(\gamma-k)}dy \\ &= 2^{-\frac{d}{2}}e^{-2\pi i\frac{1}{2}u\cdot(\gamma-k)}\int\varphi(y)\,e^{-2\pi iy\cdot\frac{1}{2}(\gamma-k)}dy = 2^{-\frac{d}{2}}e^{-2\pi iu\cdot\frac{1}{2}(\gamma-k)}\widehat{\varphi}\left(\frac{1}{2}(\gamma - k)\right).\end{aligned}$$

(*d*) We now compute $\sigma(V_1)$. For $\mathbf{c} \in l^2(\mathbb{Z}^d)$ define

$$\mathbf{c}_u(k) = \begin{cases} \mathbf{c}(k) & \text{if} \quad k = 2m+u,\ m \in \mathbb{Z}^d, \\ 0 & \text{otherwise.} \end{cases}$$

Two multi-integers m, n are congruent mod $\{0,1\}^d$ if they have the same multi-remainder $u \in \{0,1\}^d$, i.e., if $m = 2k_1 + u$ and $n = 2k_2 + u$ where $k_1, k_2 \in \mathbb{Z}^d$ and $u \in \{0,1\}^d$. It follows that $\mathbf{c} = \sum_{u \in \{0,1\}^d} \mathbf{c}_u$ and

$$\langle \mathbf{c}_u, \mathbf{c}_v \rangle = \delta(u,v) \|\mathbf{c}_u\|^2,$$

where δ is the Kronecker delta. Hence,

$$\begin{aligned} \left(e^{-2\pi i u \cdot \frac{1}{2}(\gamma-k)} \widehat{\varphi}\left(\frac{1}{2}(\gamma-k)\right)\right)_{k\in\mathbb{Z}^d} &= \left(e^{\pi i u\cdot k} e^{-\pi i u\cdot\gamma} \widehat{\varphi}\left(\frac{1}{2}(\gamma-k)\right)\right)_{k\in\mathbb{Z}^d} \\ &= e^{-\pi i u\cdot\gamma} \sum_{v\in\{0,1\}^d} e^{\pi i u\cdot v} \mathbf{c}_{v,\gamma}, \end{aligned}$$

where $\mathbf{c}_\gamma(k) = \widehat{\varphi}\left(\frac{1}{2}(\gamma-k)\right)$ and $\mathbf{c}_{v,\gamma}$ is defined as above (only the v-congruent entries survive). Therefore, $\mathcal{X}(V_1)(\gamma) = \operatorname{span}\left\{ \mathbf{c}_{u,\gamma} : u \in \{0,1\}^d \right\}$ and

$$\sigma(V_1) = \left\{\gamma \in \mathbb{T}^d : \dim \mathcal{X}(V_1)(\gamma) > 0\right\} = \left\{\gamma \in \mathbb{T}^d : \text{ at least one } \mathbf{c}_{u,\gamma} \neq 0\right\}.$$

On the other hand $\mathbf{c}_{u,\gamma} \neq 0$ implies that $\|\mathbf{c}_\gamma\|^2 = \sum_{u\in\{0,1\}^d} \|\mathbf{c}_{u,\gamma}\|^2 > 0$ by the Pythagorean theorem. We compute

$$\begin{aligned} \sigma(V_1) &= \left\{\gamma \in \mathbb{T}^d : \|\mathbf{c}_\gamma\|^2 > 0\right\} \\ &= \left\{\gamma \in \mathbb{T}^d : \sum_{k\in\mathbb{Z}^d} \left|\widehat{\varphi}\left(\frac{\gamma}{2} - \frac{2k+u}{2}\right)\right|^2 > 0, \text{ for some } u\right\} \\ &= \left\{\gamma \in \mathbb{T}^d : \sum_{k\in\mathbb{Z}^d} \left|\widehat{\varphi}\left(\frac{\gamma}{2} - \frac{u}{2} + k\right)\right|^2 > 0, \text{ for some } u\right\} \\ &= \left\{\gamma \in \mathbb{T}^d : \Phi\left(\frac{\gamma}{2} - \frac{u}{2}\right) > 0, \text{ for some } u\right\} \\ &= \left\{\gamma \in \mathbb{T}^d : \Phi\left(\frac{\gamma}{2} + \frac{u}{2}\right) > 0, \text{ for some } u\right\}. \end{aligned}$$

If we now define $H_{0,u} = H_0\left(\frac{\gamma}{2} - \frac{u}{2}\right)$, then,

$$\left(H_0\left(\frac{1}{2}(\gamma-k)\right)\widehat{\varphi}\left(\frac{1}{2}(\gamma-k)\right)\right)_{k\in\mathbb{Z}^d} = \sum_{u\in\{0,1\}^d} H_{0,u}\mathbf{c}_{u,\gamma}$$

because H_0 is 1-periodic in each variable. Hence,

$$\mathcal{X}(V_0)(\gamma) = \operatorname{span}\left\{ \sum_{u\in\{0,1\}^d} H_{0,u}\mathbf{c}_{u,\gamma} \right\} \subset \operatorname{span}\left\{\mathbf{c}_{u,\gamma} : u \in \{0,1\}^d\right\} = \mathcal{X}(V_1)(\gamma).$$

(*e*) $\Gamma = \frac{1}{2}\left\{\gamma \in \mathbb{T}^d : \dim \mathcal{X}(V_1)(\gamma) = 2^d \text{ and } \forall u \in \{0,1\}^d,\ H_{0,u} = 0\right\}$. If we define $\Gamma_j := \left\{\gamma \in \mathbb{T}^d : D_{V_1}(\gamma) = \dim \mathcal{X}(V_1)(\gamma) = j.\right\}, 1 \le j \le 2^d$, then,

$$\sigma(V_1) = \bigcup_j \Gamma_j,$$

a disjoint union. If $\gamma \in \Gamma_{2^d}$ and $H_{0,u} = 0$ for every $u \in \{0,1\}^d$, then we have that $\mathcal{X}(V_0)(\gamma) = \{0\}$ and $\mathcal{X}(W_0)(\gamma) = \mathcal{X}(V_1)(\gamma)$. Hence, $D_{W_0}(\gamma) = \dim \mathcal{X}(W_0)(\gamma) = 2^d$, since $D_{V_1}(\gamma) = \dim \mathcal{X}(V_1)(\gamma) = 2^d$ $(\gamma \in \Gamma_{2^d})$. Now,

$$\begin{aligned}
\Theta &= \left\{\gamma \in \Gamma_{2^d} : \forall u \in \{0,1\}^d,\ H_{0,u} = 0\right\} \\
&= \left\{\gamma \in \Gamma_{2^d} : \forall u \in \{0,1\}^d,\ H_0\left(\frac{\gamma}{2} - \frac{u}{2}\right) = 0\right\} \\
&= \left\{\gamma \in \Gamma_{2^d} : \forall u \in \{0,1\}^d,\ H_0\left(\frac{\gamma}{2} + \frac{u}{2}\right) = 0\right\} \\
&= \left\{\gamma \in \mathbb{T}^d : \forall u \in \{0,1\}^d,\ \Phi(\gamma) = 0, \Phi\left(\frac{\gamma}{2} + \frac{u}{2}\right) > 0\right\} \\
&= \left\{2\lambda \in \mathbb{T}^d : \forall u \in \{0,1\}^d,\ \Phi(2\lambda) = 0, \Phi\left(\lambda + \frac{u}{2}\right) > 0\right\} = 2\Gamma.
\end{aligned}$$

Hence, $|\Theta| = 2^d |\Gamma|$, which implies that $|\Theta| > 0$ if and only if $|\Gamma| > 0$. It is now clear that if $|\Gamma| > 0$, then $D_{W_0}(\gamma) = 2^d$ in a subset of $\mathbb{T}^d$ with positive measure. Further, because of the way Θ is defined, *at least* 2^d *wavelets are necessary*, since in this case, len $W_0 = 2^d$. Now, applyng Theorems 1, 2, and 3 concerning shift-invariant subspaces, we obtain the result. □

Remark 7. *Equation (4.6),*

$$\Phi(2\gamma) = \sum_{u \in \{0,1\}^d} |H_0|^2\left(\gamma + \frac{u}{2}\right)\Phi\left(\gamma + \frac{u}{2}\right),$$

is true a.e. The left side of this equation is zero in Γ, *and* $\Phi\left(\gamma + \frac{u}{2}\right) > 0$ *in* Γ. *Hence,* $H_0\left(\gamma + \frac{u}{2}\right) = 0$, *i.e.,*

$$\Gamma \subset \left\{\gamma \in \mathbb{T}^d : \forall u \in \{0,1\}^d,\ H_0\left(\gamma + \frac{u}{2}\right) = 0\right\}.$$

In other words, Γ *can be seen as a subset of the zero set of the low pass filter* H_0 *with the additional geometric condition that its elements* γ *also have the property that* $\gamma + \frac{u}{2}$ *is a zero of* H_0. *This observation can also be obtained from the proof of the previous theorem (see part (e) in the proof of Theorem 9).*

Because of the conclusions of Theorem 8 and Theorem 9, it seems reasonable to point out the distinction between these two results. Theorem 8 provides sufficient conditions in terms of equations (4.5), (4.7), and (4.12) developed in section 4, that a finite sequence of elements is a frame of W_0. Theorem 9 gives a lower bound for the number of generators by a measure theoretic criterion. Theorem 9 is an existence (nonconstructive) theorem and Theorem 8 can be viewed as a recipe to give explicitly the generators.

Remark 8. *Inspired by the work of Ron and Shen, there is a form of the Unitary Extension Principle in* [9]. *In this case, the matrix equation*

$$\mathbf{H}(\gamma)\mathbf{H}(\gamma)^* = I_{2^d \times 2^d} \text{ for almost every } \gamma \text{ in } \sigma(V_0)$$

forces

$$m \ge 2^d - 1.$$

This follows from the following argument:

$$2^d = rank\left(I_{2^d\times 2^d}\right) = rank\left(\mathbf{H}(\gamma)\,\mathbf{H}(\gamma)^*\right)$$
$$\leq rank\left(\mathbf{H}(\gamma)^*\right) = rank\left(\mathbf{H}(\gamma)\right) \leq m+1 = \text{the number of columns of } \mathbf{H}(\gamma).$$

Hence, the number of wavelets is at least $2^d - 1$. *Hence, the number of wavelet generators grows exponentially with the dimension* d, *as in the case of dyadic multiresolution analysis. On the other hand, the matrix equation (4.12) implies that a lower bound for the number of wavelet generators can be any number bigger than or equal to* 1, *depending how many entries of the diagonal in the right side of (4.12) are nonzero. This gives the possibility of constructing single wavelets in higher dimensions (see Examples 1 and 2), a feature not present in the UEP.*

Example 3. *If the scaling function is defined by*

$$\widehat{\varphi}(\gamma) = \mathbf{1}_{\{\|\gamma\|\leq a\}}(\gamma),$$

where $1/4 < a < 1/2$, *then the set* $\Gamma = \left\{\gamma\in\mathbb{T}^d : \Phi(2\gamma) = 0,\ \Phi\left(\gamma+\frac{u}{2}\right) > 0,\ u\in\{0,1\}^d\right\}$ *has positive measure. Hence, in view of the previous result, at least* 2^d *wavelets are needed to generate a wavelet frame for* $L^2(\mathbb{R}^d)$. *In this case, a set* $\{\psi_u\}$ *of wavelets for* $L^2(\mathbb{R}^d)$ *is given by*

$$\widehat{\psi_u}(\gamma) = 2^{-d/2}e^{-\pi i u\cdot\gamma}\left[\mathbf{1}_{\{\|\gamma\|\leq 2a\}}(\gamma) - \mathbf{1}_{\{\|\gamma\|\leq a\}}(\gamma)\right],\ u\in\{0,1\}^d.$$

These radial wavelet frames have several recent applications, e.g., for texture detection in medical imaging [11, 35].

6. Algorithm

The following result allows one to construct a frame of translates in higher dimensions by means of tensor products. It is a special case of a theorem found in [22], and it can be proved directly by a straightforward calculation.

Proposition 1. *Let* $\{\tau_k\varphi\}_{k\in\mathbb{Z}}$ *and* $\{\tau_k\varphi'\}_{k\in\mathbb{Z}}$ *be two frames for* $L^2(\mathbb{R})$ *with frame bounds* $0 < A \leq B < \infty$ *and* $0 < A' \leq B' < \infty$ *respectively, then* $\{\tau_m(\varphi\otimes\varphi')\}_{m\in\mathbb{Z}^2}$ *is a frame for* $L^2\left(\mathbb{R}^2\right)$ *with frame bounds* AA' *and* BB'.

Following Proposition 1, we shall construct FMRA wavelets by tensor products in the same way it is done for the classical MRA case. First, assume that (V_j,φ) is an FMRA of $L^2(\mathbb{R})$. Assume the set Γ defined by

$$\Gamma = \left\{\gamma\in\mathbb{T} : \Phi(2\gamma) = 0,\ \Phi(\gamma) > 0,\ \Phi\left(\gamma+\frac{1}{2}\right) > 0\right\}$$

has measure zero, and the wavelet ψ is given by

$$\widehat{\psi}(2\gamma) = H_1(\gamma)\,\widehat{\varphi}(\gamma),$$

where $H_1(\gamma)$ is defined by

$$H_1(\gamma) = \begin{cases} e^{-2\pi i\gamma}\overline{H_0}\left(\gamma+\frac{1}{2}\right)\Phi\left(\gamma+\frac{1}{2}\right) & \text{if} \quad \gamma\in\Delta_2, \\ 1 & \text{if} \quad \gamma\in\Delta_3 \text{ and } H_0(\gamma) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The sets Δ_j, $j = 1, 2, 3, 4$, are

$$\Delta_1 = \left\{\gamma\in\mathbb{T} : \Phi(\gamma) = 0,\ \Phi\left(\gamma + \frac{1}{2}\right) = 0\right\},$$
$$\Delta_2 = \left\{\gamma\in\mathbb{T} : \Phi(\gamma) > 0 \text{ and } \Phi\left(\gamma + \frac{1}{2}\right) > 0\right\},$$
$$\Delta_3 = \left\{\gamma\in\mathbb{T} : \Phi(\gamma) > 0 \text{ and } \Phi\left(\gamma + \frac{1}{2}\right) = 0\right\},$$
$$\Delta_4 = \left\{\gamma\in\mathbb{T} : \Phi(\gamma) = 0 \text{ and } \Phi\left(\gamma + \frac{1}{2}\right) > 0\right\},$$

and they form a partition of $\mathbb{T}$, see [9, 17]. We now define the d-fold tensor product $V_1^{(d)} = \bigotimes_d V_1$ associated with the given FMRA of $L^2(\mathbb{R})$. Recall that

$$V_1 = V_0 \bigoplus W_0 \subset L^2(\mathbb{R}).$$

Hence, if we set $X_0 = V_0$, $X_1 = W_0$, and

$$X_\nu = X_{\nu_1} \bigotimes X_{\nu_2} \bigotimes ... \bigotimes X_{\nu_1},$$

for $\nu = (\nu_1, ..., \nu_d) \in \{0, 1\}^d$, we have

$$\bigotimes_d V_1 = \bigoplus_{\nu\in\{0,1\}^d} X_\nu.$$

With the convention that $\psi_0 = \varphi$, denote

$$\psi_\nu(x_1, ..., x_d) = \psi_{\nu_1}(x_1)\,\psi_{\nu_2}(x_2)\,...\psi_{\nu_d}(x_d).$$

By Lemma 1, $\{\tau_k\psi_\nu\}_{k\in\mathbb{Z}^d}$ is a frame for X_ν, so that $\{\tau_k\psi_\nu\}_{k\in\mathbb{Z}^d,\nu\in\{0,1\}^d}$ is a frame for $V_1^{(d)}$.

In order to write these wavelets as a Mallat-Meyer algorithm, we compute $\Phi^{(d)}$ obtained by the periodization of the square of the modulus of $\widehat{\varphi}(\gamma_1)...\widehat{\varphi}(\gamma_d)$:

$$\begin{aligned}
\Phi^{(d)}(\gamma_1, ..., \gamma_d) &= \sum_{(k_1,...,k_d)\in\mathbb{Z}^d} |\widehat{\varphi}(\gamma_1 + k_1)...\widehat{\varphi}(\gamma_d + k_d)|^2 \\
&= \sum_{(k_1,...,k_d)\in\mathbb{Z}^d} |\widehat{\varphi}(\gamma_1 + k_1)|^2 ... |\widehat{\varphi}(\gamma_d + k_d)|^2 \\
&= \sum_{k_1\in\mathbb{Z}} ... \sum_{k_d\in\mathbb{Z}} |\widehat{\varphi}(\gamma_1 + k_1)|^2 ... |\widehat{\varphi}(\gamma_d + k_d)|^2 \\
&= \left[\sum_{k_1\in\mathbb{Z}} |\widehat{\varphi}(\gamma_1 + k_1)|^2\right] ... \left[\sum_{k_d\in\mathbb{Z}} |\widehat{\varphi}(\gamma_d + k_d)|^2\right] \\
&= \Phi(\gamma_1)...\Phi(\gamma_d).
\end{aligned}$$

We then define Φ_ν by $\Phi_\nu(\gamma_1, ..., \gamma_d) = \Phi_{\nu_1}(\gamma_1)...\Phi_{\nu_d}(\gamma_d)$, where $\Phi_{(0,...,0)}(\gamma_1, ..., \gamma_d) = 0$,

$$\Phi_{\nu_j}(\gamma_j) = \begin{cases} 1 & \text{if} \quad \nu_j = 0 \\ \Phi(\gamma_j) & \text{if} \quad \nu_j = 1, \end{cases}$$

and H^ν by $H^\nu(\gamma_1, ..., \gamma_d) = H^{\nu_1}(\gamma_1)...H^{\nu_d}(\gamma_d)$, where

$$H^{\nu_j}(\gamma_j) = \begin{cases} \overline{H_0}(\gamma_j) & \text{if} \quad \nu_j = 1 \\ H_0(\gamma_j) & \text{if} \quad \nu_j = 0. \end{cases}$$

The sets $\Delta_j^{(d)}$, $j = 1, 2, 3, 4$, for the tensor product, take the form

$$\Delta_1^{(d)} = \left\{\gamma\in\mathbb{T}^d : \forall u \in \{0,1\}^d,\ \Phi^{(d)}\left(\gamma + \frac{u}{2}\right) = 0\right\},$$

$$\Delta_2^{(d)} = \left\{\gamma\in\mathbb{T}^d : \Phi^{(d)}(\gamma) > 0 \text{ and } \exists u \in \{0,1\}^d \setminus \{0\},\ \Phi^{(d)}\left(\gamma + \frac{u}{2}\right) > 0\right\},$$

$$\Delta_3^{(d)} = \left\{\gamma\in\mathbb{T}^d : \Phi^{(d)}(\gamma) > 0 \text{ and } \Phi\left(\gamma_1 + \frac{1}{2}\right) = ... = \Phi\left(\gamma_d + \frac{1}{2}\right) = 0\right\},$$

$$\Delta_4^{(d)} = \left\{\gamma\in\mathbb{T}^d : \Phi^{(d)}(\gamma) = 0 \text{ and } \exists u \in \{0,1\}^d \setminus \{0\} \text{ with } \Phi^{(d)}\left(\gamma + \frac{u}{2}\right) > 0\right\}.$$

Moreover, the filters $H_\nu(\gamma_1, ..., \gamma_d) = H_{\nu_1}(\gamma_1)...H_{\nu_d}(\gamma_d)$, for $\nu \neq (0, ..., 0)$ are given by

$$H_\nu(\gamma) = \begin{cases} e^{-2\pi i\gamma\cdot\nu} H^\nu\left(\gamma + \frac{\nu}{2}\right)\Phi_\nu\left(\gamma + \frac{\nu}{2}\right) & \text{if} \quad \gamma \in \Delta_2, \\ \prod\limits_{\nu_p=0} H_0(\gamma_p) & \text{if} \quad \gamma \in \Delta_3,\ H^{\nu_j}(\gamma_j) = 0, \nu_j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where, by convention, the product $\prod_{\nu_p=0} H_0(\gamma_p) = 1$ in the case none of the ν_p is 0, i.e., $\nu = (1, ..., 1)$. H_0 is the low pass filter on $\mathbb{T}$. Thus, the FMRA wavelets, ψ_ν, $\nu \neq (0, ..., 0)$, defined above can now be formulated as in the Mallat-Meyer algorithm as follows:

$$\widehat{\psi_\nu}(2\gamma) = H_\nu(\gamma)\,\widehat{\varphi^{(d)}}(\gamma),$$

where $\varphi^{(d)}(x_1, ..., x_d) = \varphi(x_1)...\varphi(x_d)$.

Remark 9. *This argument can be generalized to the case $(V_j(1), \varphi_1), ..., (V_j(d), \varphi_d)$ of d distinct FMRAs of $L^2(\mathbb{R})$. The idea is the same, but beginning with*

$$\Phi^{(d)}(\gamma_1, ..., \gamma_d) = \Phi_1(\gamma_1)...\Phi_d(\gamma_d).$$

However, the notation becomes cumbersome. If one of the sets

$$\Gamma_j = \left\{\gamma\in\mathbb{T} : \Phi_j(2\gamma) = 0,\ \Phi_j(\gamma) > 0,\ \Phi_j\left(\gamma + \frac{1}{2}\right) > 0\right\}$$

has measure zero, then Theorem 9 says that a set of FMRA wavelets for $L^2(\mathbb{R}^d)$, with cardinality less than or equal $2^d - 1$, exists. Moreover, if Γ_k, $k \neq j$, has positive measure, then it is impossible to construct via tensor products a set of wavelet generators with cardinality less than or equal $2^d - 1$, since we need at least two wavelet generators for $W_0(k) = V_0(k)^\perp \cap V_1(k)$ by Theorem 9.

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Two Problems from Industry and Their Solutions Via Harmonic and Complex Analysis

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Abstract

This paper focuses in on two problems – signal reconstruction from linear translation-invariant systems (deconvolution) and reconstruction from samples (sampling theory). These problems are realized in remote sensing and digital-to-analog reconstruction in industry. The tools used to solve these problems include techniques from complex function theory, Fourier and harmonic analysis, real and functional analysis, and number theory. What may be more interesting than the solutions is how the they are so strongly interrelated. Data is gathered on discrete lattices which are developed using some results from number theory. Reconstruction of the continuous information is then developed using interpolation techniques from the theory of entire functions with restricted growth[1].

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1 Introduction

Reconstruction of information from raw data is one of the most fundamental problems in science. These problems fit into two basic categories – deterministic and statistical. Deterministic problems include (but are certainly not limited to) reconstruction from sample values (sampling theory), rebuilding signals from coefficients (Fourier synthesis, wavelet reconstruction), and reconstruction from boundary or initial values assuming a given model (solving differential and integral equations). Statistical reconstruction is used when there is a random component (noise) in the information, and aims at developing either statistics about the signal or getting the best representative of the signal. These problems include parameter estimation, blind deconvolution, matched filtering, and detection and estimation.

This paper addresses deterministic reconstruction. In particular, it focuses in on two problems – reconstruction from samples (sampling theory), and signal reconstruction from linear translation-invariant systems (deconvolution). The tools used to solve these problems are drawn from a wide spectrum – complex function theory, Fourier and harmonic analysis, real and functional analysis, and even number theory. What may be more interesting than the solutions is how they are so strongly interrelated. Data is gathered on discrete lattices which are developed using some results from number theory. Reconstruction of the continuous information is then developed using interpolation techniques from the theory of entire functions with restricted growth.

We will assume definitions and theorems from harmonic and complex analysis. Much of what follows is standard notation in the literature, and can be found in texts covering those subjects. For example, see Apostol [1], Benedetto [2], Barros-Neto [3], Conway [28], Dym and McKean [31], Higgins [35], Hörmander [37], Körner [42], Levin [44], Marks [47], Young [66] and Zayed [69]. We define the Fourier transform of $f \in L^1(\mathbb{R})$ as

$$\widehat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i t\omega} dt$$

and its inversion formula, for $\widehat{f} \in L^1(\widehat{\mathbb{R}})$, as

$$f(t) = \int_{\widehat{\mathbb{R}}} \widehat{f}(\omega) e^{2\pi i \omega t} d\omega \,,$$

using the same normalization as that of Benedetto [2] and Dym and McKean [31].

1.1 Two Problems from Industry

Two key problems in signal and image processing are the recovery of original data from the procedures which gather the information (*deconvolution*), and the reconstruction of analog information

from discrete data (*sampling*). We have developed solutions to both of these problems using techniques from the theory of complex variables, and in particular the aspects of that theory which allow for interpolation from discrete data values. Deconvolution is discussed in Theorems 1-9 in the paper. The key step in our solutions of deconvolution problems is an interpolation from discrete data, given on the unions of lattices with pairwise non-commensurate generators. The deconvoluton techniques in turn provide insight into how one can develop general sampling schemes on such sets. We give specific examples of non-commensurate sampling lattices, and use a generalization of B. Ya. Levin's sine-type functions to develop interpolating formulae on these sets. Sampling results are presented in Theorems 10-12 in the paper.

The solution to the first problem involves a new approach to deconvolution.

Problem 1 : Scanning Computer Chips *Intel, Dell, Texas Instruments, ...*

The construction of new computer chips requires increasingly fine measurement tools. The measurement tools can be modeled as a convolution $s(t) = f * \mu(t) = \int f(\tau)\mu(t-\tau)\,d\tau$ where f is the input function of the surface of the chip, μ is the sensor function, and s is the measured surface. Convolution "smooths out" or "blurs" functions. It also "erases" some functions. The industry requires an increasingly accurate surface measurement s, but the physical limits in refining μ have been reached ([51]). It is possible, even given these limits on μ, to refine the measurement of s. This is achieved by *multichannel deconvolution*. We call this the *Multichannel Deconvolution Problem (MDP)*.

Mathematically, this problem can be thought of in the following way. Given a collection of compactly supported distributions, $\{\mu_i\}_{i=1}^m \subseteq \mathcal{E}'(\mathbb{R}^d)$, we want to recover an arbitrary function $f \in C^\infty(\mathbb{R}^d)$ from the data $\{s_i\}_{i=1}^m = \{f * \mu_i\}_{i=1}^m$. This is a two-step problem. The first step is to give conditions on $\{\mu_i\}_{i=1}^m$ which allow for this reconstruction. The second step involves the construction of *deconvolvers*, which come in a variety of types but which are essentially a collection of distributions which (1) depend only on the convolvers $\{\mu_i\}_{i=1}^m$ and (2) allow for the solution of the MDP with only simple linear operations on the data $\{s_i\}_{i=1}^m$.

For example, any set of distributions $\{\nu_i\}_{i=1}^m \subseteq \mathcal{D}'(\mathbb{R}^d)$ which satisfy

$$\sum_{i=1}^m \mu_i * \nu_i = \delta \tag{1}$$

or equivalently,

$$\sum_{i=1}^m \widehat{\mu_i}(\gamma)\widehat{\nu_i}(\gamma) = 1 \tag{2}$$

is a set of deconvolvers. In this case, f may be recovered by

$$\sum_{i=1}^{m} s_i * \nu_i = \sum_{i=1}^{m} (f * \mu_i) * \nu_i = \sum_{i=1}^{m} f * (\mu_i * \nu_i)$$
$$= f * \sum_{i=1}^{m} \mu_i * \nu_i = f * \delta = f\,, \tag{3}$$

provided that the associative law holds.

Equation (2) is a type of *Bezout equation*. Polynomial Bezout equations have been studied for over a century. The equation above is an *analytic Bezout equation*, in which we are dealing with transcendental entire functions, rather than polynomials. Such problems have been extensively studied in a variety of contexts, including the study of division problems, interpolation, analytic continuation, complexity theory, number theory, and solution to systems of PDE's ([6], [13], [14], [15], [36], [40]). For the purposes of this paper, we require the following result of Hörmander which gives necessary and sufficient conditions under which compactly supported solutions of (1) exist. Hörmander's result solves the first step of the MDP.

Theorem 1 [36] *There exist compactly supported distributions* $\{\nu_i\}_{i=1}^m \subseteq \mathcal{E}'(\mathbb{R}^d)$ *solving* (1) *if and only if there exist constants* A, B, $N > 0$ *such that*

$$\sum_{i=1}^{m} |\widehat{\mu_i}(z)| \geq A(1+|z|)^{-N}\, e^{-B|\Im z|} \quad \forall z \in \mathbb{C}^d. \tag{4}$$

A collection $\{\mu_i\}_{i=1}^m \subseteq \mathcal{E}'(\mathbb{R}^d)$ which satisfies (4) is said to be *strongly coprime* . It is of interest to compare the envelope condition above with the Paley-Wiener-Schwarz growth constraint. Note that Hörmander's envelope condition is essentially the inversion of this constraint.

The heart of the multichannel theory involves solving an interpolation problem, reconstructing functions (the deconvolvers) in a space of restricted growth ($\widehat{\mathcal{E}'}$) from discrete data (their values on the zero sets of the convolvers). This gives solutions to the Bezout equation [21, 26, 27]. This development utilizes the zero sets of the $\widehat{\mu_i}$ as different sampling rates. This then leads naturally to our second problem.

Problem 2 : Digital-to-Analog Conversion *Phillips, Sony, ...*

We are increasingly displaying, storing and/or analyzing signals and images in digital formats. The mathematical theories for digital-to-analog conversion are categorized under Sampling Theory. The industry requires increasingly faster sampling schemes in order to store greater amounts of information. Current sampling rates are approaching their maximum rates. It is possible to speed up the sampling schemes by sampling with multiple samplers, each operating at rates slower than

the given maximum. The samples are interlaced, and the net effect is that of a faster sampling. This procedure is called *multirate sampling*, and we call this the *Multirate Sampling Problem (MRSP).*

The connection between sampling and multichannel deconvolution leads us to sampling schemes on properly chosen non-commensurate lattices. Let $\{r_i\}_{i=1}^m$ be a set such that r_i/r_j is *poorly approximated by rationals* for $i \neq j$, and let $R = \sum_{i=1}^m r_i$. Let

$$\Lambda = \left(\bigcup_{i=1}^{m} \left\{ \frac{\pm k}{2r_i} \right\}_{k \in \mathbb{N}} \right) \bigcup \{0\}$$

for $k \in \mathbb{N}$ be a sampling grid, made up of a union of sampling grids with non-commensurate generators $\{r_i\}$. Given an R bandlimited function f, we can reconstruct f from samples taken on Λ. We give specific examples of these non-commensurate sampling lattices, and use a generalization of Levin's sine-type functions to develop interpolating formulae on these lattices. Our interpolators are products of sine-type functions. We show that our reconstruction formulae converge conditionally.

Solutions to Problem 1 and Problem 2 are developed using similar tools – complex function theory, Fourier and harmonic analysis, real and functional analysis, and even number theory. What may be more interesting than the solutions is how the they are so strongly interrelated. Data is gathered on discrete lattices which are developed using some results from number theory. Reconstruction of the continuous information is then developed using interpolation techniques from the theory of entire functions with restricted growth.

2 Multichannel Deconvolution

2.1 The Coprime Condition and Local Deconvolution

The advantage to solving (1) with compactly supported deconvolvers is that in this case, deconvolution can be done *locally.* That is, we can recover f at some point t_0 from knowledge of the data $\{s_i\}_{i=1}^m$ on some compact set possibly depending on t_0. Local deconvolution can be achieved if (1) can be solved with deconvolvers in $\mathcal{E}'(\mathbb{R}^d)$ as the following calculation shows. (Let $\widetilde{h}(t) = h(-t)$ here and below).

$$\begin{aligned} f(t_0) \;&=\; \langle \widetilde{f}, \tau_{-t_0}\delta \rangle = \left\langle \tau_{t_0}\widetilde{f},\; \sum_{i=1}^{m} \mu_i * \nu_i \right\rangle \\ &= \sum_{i=1}^{m} \langle \tau_{t_0}\widetilde{f} * \widetilde{\mu_i}, \nu_i \rangle = \sum_{i=1}^{m} \langle \tau_{t_0}\widetilde{s_i}, \nu_i \rangle. \end{aligned}$$

Thus, $f(t_0)$ can be recovered from knowledge of s_i on the compact set $\{t_0\} + \operatorname{supp} \nu_i$. Hörmander's Theorem (Theorem 1) says that local deconvolution in the sense of (1) is possible as long as the strongly coprime condition is satisfied.

In contrast, non-local deconvolution in the sense of (1) is possible trivially in the following sense. For $1 \le i \le m$, let

$$\widehat{\nu_i}(\omega) = \overline{\widehat{\mu_i}(\omega)}\,\Phi(\omega) \quad \text{where} \quad \Phi(\omega) = \left(\sum_{j=1}^{m} |\widehat{\mu_j}(\omega)|^2\right)^{-1}. \tag{5}$$

Then for all $\omega \in \widehat{\mathbb{R}}^d$,

$$\sum_{i=1}^{m} \widehat{\mu_i}(\omega)\widehat{\nu_i}(\omega) = \sum_{i=1}^{m} \widehat{\mu_i}(\omega)\overline{\widehat{\mu_i}(\omega)}\,\Phi(\omega) = \Phi(\omega)\sum_{i=1}^{m} |\widehat{\mu_i}(\omega)|^2 = 1.$$

In light of the coprime assumption, $\Phi(\omega)$ has polynomial growth in ω and hence is the Fourier transform of a tempered distribution. It follows that each ν_i is a tempered distribution. However, the functions $\widehat{\nu_i}$ are not entire. They are not analytic, but rather, anti-analytic – vanishing under the differential operator $\partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)$ instead of the Cauchy-Riemann operator $\overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$. Thus, the Paley-Wiener-Schwartz Theorem asserts that the distribution ν_i cannot be compactly supported. Hence deconvolution is not local with these deconvolvers.

An alternate approach to the MDP is to solve a modified version of (1), namely

$$\sum_{i=1}^{m} \mu_i * \nu_{i,\varphi} = \varphi \tag{6}$$

or equivalently, the modified Bezout equation

$$\sum_{i=1}^{m} \widehat{\mu_{i,\varphi}}(\omega)\widehat{\nu_i}(\omega) = \widehat{\varphi}(\omega) \tag{7}$$

where $\varphi \in C_c^k(\mathbb{R}^d)$, for a sufficient degree of smoothness k. If there exist compactly supported solutions to (1), then certainly there exist compactly supported solutions to (6), namely $\nu_{i,\varphi} = \nu_i * \varphi$. In this case, the deconvolvers are themselves in $C_c^k(\mathbb{R}^d)$. Such smooth deconvolvers have an advantage in numerical applications of the theory.

If (6) can be solved, then the MDP can be solved locally as follows. Choose $\psi \in \mathbb{C}_c^\infty(\mathbb{R}^d)$ so that $\psi(t) \ge 0$ for all t and $\int_{\mathbb{R}^d} \psi(t)\,dt = 1$. For $\lambda \ge 1$, define $\varphi(t) = \psi_\lambda(t) = \lambda^d\,\psi(\lambda t)$, i.e., $\{\psi_\lambda\}$ is an approximate identity family. Define $\{\nu_{i,\psi_\lambda}\}_{i=1}^m$ to be a solution of (6) with $\varphi(t) = \psi_\lambda(t)$. Then,

$$f = \lim_{\lambda\to\infty}(f * \psi_\lambda) = \sum_{i=1}^{m} \lim_{\lambda\to\infty}(f * \mu_i * \nu_{i,\psi_\lambda}) = \sum_{i=1}^{m} \lim_{\lambda\to\infty}(s_i * \nu_{i,\psi_\lambda})$$

with convergence in $\mathcal{E}'$. The solution is local in light of the fact that

$$f(t_0) = \lim_{\lambda\to\infty}\langle f, \tau_{t_0}\widetilde{\psi}_\lambda\rangle = \sum_{i=1}^{m}\langle \tau_{t_0}\widetilde{s}_i, \nu_{i,\psi_\lambda}\rangle.$$

Thus, recovery of f at t_0 requires knowledge of s_i on the compact set $\{t_0\} + (\operatorname{supp}\nu_i + \operatorname{supp}\psi)$.

A similar approach is the following. Let $\{\psi_n\}_{n=1}^{\infty}$ be an orthonormal basis for $L^2(\mathbb{R}^d)$. (We could also have $\{\psi_n\}_{n=1}^{\infty}$ be a Riesz basis, a frame, or a complete set for $L^2(\mathbb{R}^d)$.) For each n, define $\{\nu_{i,\psi_n}\}_{i=1}^{m}$ to be a solution to (6) with $\varphi = \psi_n$ (assuming such a solution exists). Then arbitrary $f \in L^2(\mathbb{R}^d) \cap C^{\infty}(\mathbb{R}^d)$ can be recovered by computing

$$\langle \widetilde{f}, \psi_n \rangle = \sum_{i=1}^{m} \langle \widetilde{f}, \mu_i * \nu_{i,\psi_n} \rangle = \sum_{i=1}^{m} \langle \widetilde{f} * \widetilde{\mu}_i, \nu_{i,\psi_n} \rangle = \sum_{i=1}^{m} \langle \widetilde{s}_i, \nu_{i,\psi_n} \rangle.$$

If the basis is local, such as a smooth, compactly supported Gabor frame or wavelet basis, then recovery of f is also local [26].

2.2 Example: Cubes in $\mathbb{R}^d$

An example of convolvers that admit solutions to (1) is collections of characteristic functions of cubes in $\mathbb{R}^d$. This example is of considerable practical interest and will be used in this paper to illustrate deconvolution techniques. We have Theorem 2 due to Petersen and Meisters.

Definition 1 *A real number α is poorly approximated by rationals provided that there exist constants C, $N > 0$ such that for all integers p, q,*

$$|\alpha - p/q| \geq C|q|^{-N}. \tag{8}$$

For example, quadratic irrationals of the form $\sqrt{n}$, where $n \in \mathbb{N}$ is not a perfect square, are poorly approximated by rationals.

Theorem 2 [50] *Let $0 < r_1 < \cdots < r_m$, $m \geq d+1$ satisfy r_i/r_j is poorly approximated by rationals whenever $i \neq j$. Then the collection $\{\chi_{[-r_i,r_i]^d}\}_{i=1}^{m}$ is a strongly coprime set.*

According to Hörmander's Theorem (Theorem 1) it is necessary that the convolvers $\{\chi_{[-r_i,r_i]^d}\}_{i=1}^{m}$ satisfy the strongly coprime condition for there to exist compactly supported solutions to (1), it turns out that the MDP can be solved in a slightly weaker sense if only irrationality of r_i/r_j is assumed. In particular, the following theorem holds.

Theorem 3 ([9], [33]) *Let $0 < r_1 < \cdots < r_m$, let $m = d + 1$. Then the following are equivalent.*

(a) *The collection $\{r_i\}_{i=1}^{m}$ satisfies r_i/r_j is irrational for all $i \neq j$.*

(b) *If for $1 \leq i \leq m$, $f \in L^2(\mathbb{R}^d)$ satisfies $f * \chi_{[-r_i,r_i]^d} \equiv 0$ then $f \equiv 0$.*

Note that Theorem 3 requires only that r_i/r_j be irrational and not that these ratios be poorly approximated by rationals. The conclusion is also weaker, namely that f is completely determined by its averages, i.e., the theorem gives a solution to a *Pompeiu problem* (see [6], [68]). Theorem 3 does not say that f can be recovered from its averages in a reasonable way.

There is also a local version of Theorem 3.

Theorem 4 ([10], [33]) *Let $0 < r_1 < \cdots < r_m$, let $m = d + 1$, and let $R = \sum_{i=1}^m r_i$. Then the following are equivalent.*

(a) *The collection $\{r_i\}_{i=1}^m$ satisfies r_i/r_j is irrational for $i \neq j$.*

(b) *If for $1 \leq i \leq m$, $f \in L^2[-R, R]^d$ satisfies $f * \chi_{[-r_i, r_i]^d} = 0$ on $[-R + r_i, R - r_i]^d$ then $f \equiv 0$.*

Note that Theorem 4 allows for the unique determination of f on the cube $[-R, R]^d$ from averages of f on translates of the cubes $[-r_i, r_i]^d$, $1 \leq i \leq m$ that are *completely contained in the larger cube* $[-R, R]^d$. Again, it is not known in general how to stably recover f from these local averages. Another example of deconvolution from cubes is due to Berenstein and Yger ([11],[13]).

Theorem 5 *The family $\{\mu_1, \mu_2, \mu_3\}$ where $\mu_1 = \chi_{[0,1]^2}$, μ_2 is a rotation of μ_1 by 36° and μ_3 is a rotation of μ_1 by 45° is a strongly coprime family.*

2.3 Example: Balls in $\mathbb{R}^d$

A strongly coprime system based on characteristic functions of disks is due to Berenstein and Yger [6], [11], [13].

Theorem 6 *Let μ_1 and μ_2 be the characteristic functions of the disks $B(0, r_1)$ and $B(0, r_2) \subseteq \mathbb{R}^2$ respectively. Then the system $\{\mu_1, \mu_2\}$ is strongly coprime if and only if there is a constant $A > 0$ such that*

$$|r_1/r_2 - \xi/\eta| \geq (1/A)\,|\eta|^{-A} \tag{9}$$

for any pair $\xi, \eta > 0$ with $J_1(\xi) = J_1(\eta) = 0$ where J_1 is the Bessel function of order 1.

Explicit formulas for the construction of deconvolvers satisfying (1) are given in [11]. A local version of this theorem is also due to Berenstein and Yger [12].

Theorem 7 *Given $d \in \mathbb{N}$, let*

$$Z_d = \{\xi/\eta \colon \xi, \eta > 0,\ J_{d/2}(\xi) = J_{d/2}(\eta) = 0\},$$

let $r_1, r_2 > 0$ and let $R > r_1 + r_2$. If $r_1/r_2 \notin Z_d$, then any function f continuous on the ball $B(0,R) \subseteq \mathbb{R}^d$ is completely determined by $f * \chi_{B(0,r_1)}(t)$ for $x \in B(0, R-r_1)$ and $f * \chi_{B(0,r_2)}(t)$ for $x \in B(0, R-r_2)$.

Note that Theorem 7 allows for the unique determination of f on the ball $B(0,R)$ from averages of f on translates of the balls $B(0,r_i)$, $i = 1, 2$ that are *completely contained in the larger ball* $B(0,R)$. Explicit formulas for the construction of deconvolvers in the spirit of (6) are given in [6].

While Hörmander's Theorem (Theorem 1) supplies necessary and sufficient conditions for the solution to the MDP locally via (1), it does not supply any explicit formulas for constructing deconvolvers. Finding such explicit formulas has been the focus of much recent work ([6], [11], [13],[21], [26], [33], [59], [60], [61]). The next section develops these formulas by solving the Bezout equation using the Cauchy residue calculus.

2.4 Complex Interpolation Methods for Deconvolution

In this section, we will describe a specific solution to a modified analytic Bezout equation. The technique of solution involves the Jacobi interpolation formula and the Cauchy residue calculus. We choose $\psi \in \mathbb{C}_c^\infty(\mathbb{R})$ so that $\psi(t) \geq 0$ for all t and $\int_{\mathbb{R}} \psi(t)\, dt = 1$. For $\lambda \geq 1$, define $\varphi(t) = \psi_\lambda(t) = \lambda\, \psi(\lambda t)$, i.e., $\{\psi_\lambda\}$ is an approximate identity family.

We create an approximate deconvolution system by solving the Bezout equation

$$\sum_{i=1}^{m} \widehat{\mu}_i \cdot \widehat{\nu}_{i,\varphi} = \widehat{\varphi}.$$

Thus, $\widehat{\psi}_\lambda \longrightarrow 1$ as $\lambda \longrightarrow \infty$. This gives us our deconvolving functions. Then, as $\varphi \longrightarrow \delta$, $f * \varphi \longrightarrow f$ in the sense of convergence in $\mathcal{E}'$. That is, the signals s_i are filtered by the $\nu_{i,\varphi}$ (which have been created digitally, optically, etc., in coordination with the creation of the system and possibly tailored to be optimized under some constraint) and added, resulting in the reconstruction of $f * \varphi$. By controlling λ, f can be reconstructed to any predetermined accuracy.

Let

$$\left\{ \chi_{[-r_i, r_i]} \right\}, \; i = 1, \ldots, m \, ,$$

where $\{r_i\}_{i=1}^m$ is such that (r_i/r_j) is poorly approximated by rationals for $i \neq j$. We first note that the Fourier-Laplace transform of $\mu_i(t) = \chi_{[-r_i, r_i]}(t)$ is

$$\widehat{\mu}_i(\zeta) = \frac{\sin(2\pi r_i \zeta)}{\pi \zeta} \, .$$

As noted in the previous section, the strongly coprime condition reduces to the ratio (r_i/r_j) being poorly approximated by rationals. The optimal case when $N = 2$ is given by quadratic irrationals.

For example, if $\{p_i\}_{i=1}^{m-1}$ is a set of primes, $\{1, \sqrt{p_1}, \sqrt{p_1 p_2}, \ldots, \sqrt{p_1 p_2 \cdots p_{m-1}}\}$ will work. Let $R = \sum r_i$.

We will assume, for technical reasons, that $f \in C^\infty \cap L^2(\mathbb{R})$. We will use a density argument at the end of this section to extend the result to all of L^2. We choose φ to be a C^∞ function with support in $[-A, A]$ with $0 < A < R$ such that $\varphi \geq 0$ and $\int_{\mathbb{R}} \varphi(t)\, dt = 1$ (φ is an approximate identity). We refer to φ as the *auxiliary function* of the construction.

Let

$$\langle f, \varphi \rangle = \int_{\mathbb{R}} f(t)\, \overline{\varphi(t)}\, dt$$

be the L^2 inner product of f and φ, and let

$$\langle \nu; \varphi \rangle = \int_{\mathbb{R}} \varphi(t)\, \nu(t)\, dt$$

formally denote the dual product between $\varphi \in \mathcal{E}$ and $\nu \in \mathcal{E}'$. Let

$$\Lambda_k = \left\{ \frac{\pm n}{2r_k} \right\} \,,\ n \in \mathbb{N}$$

be the zero set of $\widehat{\mu_k}(\zeta)$.

Let's first solve the problem on

$$\Lambda = \bigcup_{i=1}^{m} \Lambda_i \,.$$

Let σ be any permutation of $\{1, 2, \ldots, m\}$ such that $\sigma(i) \neq i$. We want solutions to

$$\sum_{i=1}^{m} \widehat{\mu}_i(z) \widehat{\nu}_{i,\varphi}(z) = \widehat{\varphi}(z) \tag{10}$$

for $z \in \Lambda$ of the form

$$\widehat{\nu}_{i,\varphi}(z) = \widehat{\alpha}_{\sigma(i),\varphi}(z) \prod_{j \neq \sigma(i)} \widehat{\mu}_j(z) \,. \tag{11}$$

Substituting back and rearranging, we get

$$\sum_{i=1}^{m} \widehat{\alpha}_{i,\varphi}(z) \prod_{j \neq i} \widehat{\mu}_j(z) = \widehat{\varphi}(z) \,. \tag{12}$$

Now, fix k, and let $z \in \Lambda_k$. Then, if $i \neq k$, $\prod_{j \neq i} \widehat{\mu}_j(z) = 0$ for all $z \in \Lambda_k$. Therefore, the only non-zero term in (12) is the $i = k$ term. Thus, the solution to (12) is

$$\widehat{\alpha}_{k,\varphi}(z) = \frac{\widehat{\varphi}(z)}{\prod_{j \neq k} \widehat{\mu}_j(z)} \,. \tag{13}$$

Substituting back into (11) gives

$$\widehat{\nu}_{k,\varphi} = \widehat{\alpha}_{\sigma(k),\varphi} \prod_{j \neq \sigma(k)} \widehat{\mu}_j \,. \tag{14}$$

Thus, we can completely solve the Bezout equation on Λ. The question now becomes one of interpolation. In this section, we will use the Jacobi interpolation formula and the Cauchy residue calculus to do this interpolation.

Remark 1 For each $\widehat{\nu}_{k,\varphi}$, the zero set Λ_k is a sampling lattice. Because the sampling rates are non-commensurate, Λ is then a union of these non-commensurate sampling grids. This idea is what led to the exploration of these grids for general sampling problems.

Theorem 8 *Let $\{r_i\}_{i=1}^m$ be a set such that (r_i/r_j) is poorly approximated by rationals for $i \neq j$. Then*

$$\left\{ \chi_{[-r_i,r_i]} \right\}, \ i = 1, \ldots, m$$

is a set of strongly coprime convolvers. Let $R = \sum r_i$, and let φ be a C^∞ function with support in $[-A, A]$ with $0 < A < R$ such that $\varphi(t) \geq 0$ for all t and $\int_{\mathbb{R}} \varphi(t)\, dt = 1$. Let σ be any permutation of $\{1, 2, \ldots, m\}$ such that $\sigma(i) \neq i$, and let

$$\widehat{\alpha}_{k,\varphi}(\zeta) = \sum_{z \in \Lambda_k} \frac{\widehat{\varphi}(z)}{\prod_{j \neq k} \widehat{\mu_j}(z)} \left(\frac{1}{\frac{d}{d\zeta} \widehat{\mu_k}(z)} \right) \left(\frac{\widehat{\mu_k}(\zeta)}{\zeta - z} \right). \tag{15}$$

Given any $f \in L^2$, the deconvolvers $\nu_{i,\varphi}$ such that

$$f * \varphi = \sum_{i=1}^{m} f * \mu_i * \nu_{i,\varphi}$$

are given in the transform domain by the formulae

$$\widehat{\nu}_{k,\varphi}(\zeta) = \widehat{\alpha}_{\sigma(k),\varphi}(\zeta) \prod_{j \neq \sigma(k)} \widehat{\mu_j}(\zeta). \tag{16}$$

Remark 2 (a) We have that for $\varphi(t) = \psi_\lambda(t) = \lambda\, \psi(\lambda t)$, $\varphi(t) \longrightarrow \delta$ as $supp(\varphi) \longrightarrow \{0\}$ and so $\widehat{\varphi}(\zeta) \longrightarrow 1$ as $supp(\varphi) \longrightarrow \{0\}$ in the sense of distributions. The function φ can be chosen arbitrarily close to the δ. The advantage of constructing φ instead of the δ is that we can express our formulae for the deconvolvers as functions, and not as distributions.

(b) The formulae for the ν_i are a type of Lagrange interpolation, where the analytic functions $\widehat{\nu_i}(\zeta)$ were constructed from known values on a discrete set of data. The Cauchy residue theory is used to convert these discrete values to analytic functions. The formulae are also related to Shannon Sampling , with the functions

$$\frac{1}{\frac{d}{d\zeta} \widehat{\mu_k}(z)} \left(\frac{\widehat{\mu_k}(\zeta)}{\zeta - z} \right)^\vee (t)$$

acting as interpolators. These relationships will be explored further in the next section.

Proof : The fact that

$$\left\{\chi_{[-r_i,r_i]}\right\},\ i=1,\ldots,m$$

is strongly coprime is given by Theorem 2. Now, we have that $\widehat{\mu_k}(\zeta) = \frac{\sin(2\pi r_k\zeta)}{\pi\zeta}$. Note that $\frac{d}{d\zeta}\widehat{\mu_k}(\zeta) \neq 0$ for $\zeta \in \Lambda_k$, and that

$$\left|\frac{d}{d\zeta}\widehat{\mu_k}(\zeta)\right| = \frac{2r_k}{|\zeta|} \text{ for } \zeta \in \Lambda_k\,. \tag{17}$$

The strongly coprime condition allows us to construct a sequence of concentric circles $\{\Gamma_n\}$ with centers at the origin and radii ρ_n such that $\lim_{n\to\infty}\rho_n = \infty$ and such that there exist constants $c_k > 0$ with

$$|\sin(2\pi r_k\zeta)| \geq c_k e^{2\pi r_k|\Im\zeta|} \tag{18}$$

for $\zeta \in \Gamma_n$ (see [11]). As $\widehat{\varphi}(\zeta)$ is entire, we may represent $\widehat{\varphi}$ by the Cauchy integral formula, i.e.,

$$\widehat{\varphi}(\zeta) = \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{\widehat{\varphi}(z)}{z-\zeta}\,dz.$$

We need the following.

Lemma 1 *Let σ be any permutation of $\{1,2,\ldots,m\}$ such that $\sigma(i)\neq i$, and let*

$$\widehat{\alpha}_{k,\varphi}(\zeta) = \sum_{z\in\Lambda_k}\frac{\widehat{\varphi}(z)}{\prod_{j\neq k}\widehat{\mu_j}(z)}\left(\frac{1}{\frac{d}{d\zeta}\widehat{\mu_k}(z)}\right)\left(\frac{\widehat{\mu_k}(\zeta)}{\zeta-z}\right). \tag{19}$$

For $\zeta \in \{\zeta : |\zeta| < \rho_n\}$ and $z \in \Lambda_k$ such that $|z| < \rho_n$,

$$\widehat{\varphi}(\zeta) = \sum_{k=1}^{m}\widehat{\mu_k}(\zeta)\widehat{\alpha}_{\sigma(k),\varphi}(\zeta)\prod_{j\neq\sigma(k)}\widehat{\mu_j}(\zeta) + R_n(\zeta), \tag{20}$$

where $R_n(\zeta) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof : For $|\zeta| < \rho_n$ and $z \in \Lambda_k$ such that $|z| < \rho_n$, the Jacobi interpolation formula gives

$$\begin{aligned}
\widehat{\varphi}(\zeta) &= \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{\widehat{\varphi}(z)}{z-\zeta}\,dz \\
&= \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{\widehat{\varphi}(z)\prod_{j=1}^m\widehat{\mu_j}(z) - \widehat{\varphi}(z)\prod_{j=1}^m\widehat{\mu_j}(\zeta)}{(z-\zeta)\left(\prod_{j=1}^m\widehat{\mu_j}(z)\right)}\,dz \\
&\quad + \frac{1}{2\pi i}\prod_{j=1}^m\widehat{\mu_j}(\zeta)\oint_{\Gamma_n}\frac{\widehat{\varphi}(z)}{(z-\zeta)\left(\prod_{j=1}^m\widehat{\mu_j}(z)\right)}\,dz \\
&= \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{\widehat{\varphi}(z)\left[\prod_{j=1}^m\widehat{\mu_j}(z) - \prod_{j=1}^m\widehat{\mu_j}(\zeta)\right]}{(z-\zeta)\left(\prod_{j=1}^m\widehat{\mu_j}(z)\right)}\,dz \\
&\quad + \frac{1}{2\pi i}\prod_{j=1}^m\widehat{\mu_j}(\zeta)\oint_{\Gamma_n}\frac{\widehat{\varphi}(z)}{(z-\zeta)\left(\prod_{j=1}^m\widehat{\mu_j}(z)\right)}\,dz\,.
\end{aligned} \tag{21}$$

Now, for $z \in \Lambda_k$ for any k, $\left(\prod_{j=1}^m \widehat{\mu_j}(z)\right) = 0$, but

$$\frac{d}{d\zeta}\left(\prod_{j=1}^m \widehat{\mu_j}(z)\right) = \sum_{i=1}^m \frac{d}{d\zeta}\widehat{\mu}_i(z)\prod_{j\neq i}\widehat{\mu_j}(z) = \frac{d}{d\zeta}\widehat{\mu}_k(z)\prod_{j\neq k}\widehat{\mu_j}(z) \neq 0\,.$$

Thus, the function

$$\left(\frac{1}{\prod_{j=1}^m \widehat{\mu_j}(z)}\right)$$

has simple poles at $z \in \Lambda = \bigcup \Lambda_k$. Therefore,

$$\frac{\widehat{\varphi}(z)\left[\prod_{j=1}^m \widehat{\mu_j}(z) - \prod_{j=1}^m \widehat{\mu_j}(\zeta)\right]}{(z-\zeta)\left(\prod_{j=1}^m \widehat{\mu_j}(z)\right)}$$

has simple poles in $\Lambda = \bigcup \Lambda_k$, and so by the Cauchy Residue Theorem,

$$\begin{aligned}
&\oint_{\Gamma_n} \frac{\widehat{\varphi}(z)\left[\prod_{k=1}^m \widehat{\mu}_k(z) - \prod_{k=1}^m \widehat{\mu}_k(\zeta)\right]}{(z-\zeta)\left(\prod_{k=1}^m \widehat{\mu}_k(z)\right)}\,dz \\
&= \sum_{k=1}^m \prod_{j\neq k}\widehat{\mu_j}(\zeta)\sum_{z\in\Lambda_k}\frac{\widehat{\varphi}(z)}{\prod_{j\neq k}\widehat{\mu_j}(z)}\left(\frac{1}{\frac{d}{d\zeta}\widehat{\mu_k}(z)}\right)\left(\frac{\widehat{\mu_k}(\zeta)}{\zeta - z}\right). \qquad (22)
\end{aligned}$$

This is also valid if indices of the transforms of the convolvers are permuted, i.e.,

$$\begin{aligned}
&\sum_{k=1}^m \prod_{j\neq k}\widehat{\mu_j}(\zeta)\sum_{z\in\Lambda_k}\frac{\widehat{\varphi}(z)}{\prod_{j\neq k}\widehat{\mu_j}(z)}\left(\frac{1}{\frac{d}{d\zeta}\widehat{\mu_k}(z)}\right)\left(\frac{\widehat{\mu_k}(\zeta)}{\zeta - z}\right) \\
&= \sum_{k=1}^m \widehat{\mu_k}(\zeta)\prod_{j\neq\sigma(k)}\widehat{\mu_j}(\zeta)\sum_{z\in\Lambda_{\sigma(k)}}\frac{\widehat{\varphi}(z)}{\prod_{j\neq\sigma(k)}\widehat{\mu_j}(z)}\left(\frac{1}{\frac{d}{d\zeta}\widehat{\mu_{\sigma(k)}}(z)}\right)\left(\frac{\widehat{\mu_{\sigma(k)}}(\zeta)}{\zeta - z}\right). \qquad (23)
\end{aligned}$$

Let

$$R_n(\zeta) = \frac{1}{2\pi i}\prod_{j=1}^m \widehat{\mu_j}(\zeta)\oint_{\Gamma_n}\frac{\widehat{\varphi}(z)}{(z-\zeta)\left(\prod_{j=1}^m \widehat{\mu_j}(z)\right)}\,dz\,.$$

To finish, we need to show that $R_n(\zeta) \longrightarrow 0$ as $n \longrightarrow \infty$. We were able to choose Γ_n such that for $z \in \Gamma_n$,

$$|\sin(2\pi r_k z)| \geq c_k e^{2\pi r_k |\Im z|}\,.$$

By the Paley-Wiener-Schwartz Theorem, there exists a positive constant $C = C(N)$ such that for any positive integer N and any $z \in \mathbb{C}$,

$$|\widehat{\varphi}(z)| \leq C(1+|z|)^{-N} e^{2\pi A|\Im z|}\,.$$

Therefore, for ζ fixed and any fixed $N > 0$,

$$\begin{aligned}
|R_n(\zeta)| &\leq |\prod_{j=1}^{m} \widehat{\mu_j}(\zeta)||\frac{1}{2\pi i}| \oint_{\Gamma_n} |\frac{\widehat{\varphi}(z)}{(z-\zeta)\left(\prod_{j=1}^{m} \widehat{\mu_j}(z)\right)}|\, dz \\
&\leq |\prod_{j=1}^{m} \widehat{\mu_j}(\zeta)| \sup_{z\in\Gamma_n} \frac{1}{2\pi} |\frac{\widehat{\varphi}(z)}{(z-\zeta)\left(\prod_{j=1}^{m} \widehat{\mu_j}(z)\right)}| 2\pi\rho_n \\
&\leq |\prod_{j=1}^{m} \widehat{\mu_j}(\zeta)| \sup_{z\in\Gamma_n} \left[\frac{C(1+|z|)^{-N} e^{2\pi A|\Im z|}\rho_n}{(||z|-|\zeta||)\left(\prod_{j=1}^{m} c_j e^{2\pi r_j|\Im z|}\right)}|z|^2\right] \\
&= |\prod_{j=1}^{m} \widehat{\mu_j}(\zeta)| \left[\frac{C(1+\rho_n)^{-N}\rho_n^3}{\left(\prod_{j=1}^{m} c_j\right)(\rho_n - |\zeta|)} e^{2\pi(A-R)\rho_n}\right]. \qquad (24)
\end{aligned}$$

Since $A < R$, this last quantity $\longrightarrow 0$ as $n \longrightarrow \infty$. This completes the proof of the lemma. □

To finish the proof of the theorem, we need to show that the series expressions in the previous lemma converge in the sense of $\widehat{\mathcal{E}'}$, that is, as analytic functions which satisfy the Paley-Wiener-Schwartz growth estimate.

Lemma 2 *For $|z| < \rho_n$, the series*

$$\sum_{z\in\Lambda_k} \frac{\widehat{\varphi}(z)}{\prod_{j\neq k} \widehat{\mu_j}(z)} \left(\frac{1}{\frac{d}{d\zeta}\widehat{\mu_k}(z)}\right) \left(\frac{\widehat{\mu_k}(\zeta)}{\zeta - z}\right) \qquad (25)$$

converge in $\widehat{\mathcal{E}'}$.

Proof : If $\widehat{\mu_k}(z) = 0$, then

$$|\frac{d}{d\zeta}\widehat{\mu_k}(z)| = \frac{2r_k}{|z|}\,. \qquad (26)$$

By the arithmetic condition on (r_k/r_j), if $\widehat{\mu_k}(z) = 0$, for $j \neq k$, there exists $c_k \geq 0$ such that

$$|\widehat{\mu_j}(z)| \geq \frac{c_k}{|z|^2}\,. \qquad (27)$$

The functions $\widehat{\mu_k}(\zeta)$ are entire. Therefore, on every compact subset K of $\mathbb{C}$, there exists a constant $M_k(K)$ such that $|\widehat{\mu_k}(\zeta)| \leq M_k(K)$ for all $\zeta \in K$. The function $\varphi(t)$ is C^∞, and therefore its transform will decay rapidly on the real axis. In particular, for $z \in \Gamma$, by Paley-Wiener-Schwartz there exists a constant $d = d(3+\eta)$ such that

$$|\widehat{\varphi}(z)| \leq \frac{d}{(1+|z|)^{3+\eta}} \quad \text{for all } \eta > 0\,.$$

Therefore, for $j \neq k$,

$$\begin{aligned} & \frac{|\widehat{\varphi}(z)|}{|\prod_{j\neq k}\widehat{\mu_j}(z)\frac{d}{d\zeta}\widehat{\mu_k}(z)|}\frac{|\widehat{\mu_k}(\zeta)|}{|\zeta - z|} \\ \leq & \frac{d}{2r_k((m-1)c_k)\,(1+|z|)^{3+\eta}}|z|^3\frac{M_k(K)}{||\zeta|-|z||} \\ \leq & \frac{M_k(K)d}{2r_k((m-1)c_k)|z|^{\eta}}\frac{1}{||\zeta|-|z||}\,. \end{aligned} \tag{28}$$

In particular, for ζ fixed,

$$\frac{|\widehat{\varphi}(z)|}{|\prod_{j\neq k}\widehat{\mu_j}(z)\frac{d}{d\zeta}\widehat{\mu_k}(z)|}\frac{|\widehat{\mu_k}(\zeta)|}{|\zeta - z|} \leq \frac{(\text{constant})_k}{|z|^{1+\eta}}\,. \tag{29}$$

Thus, by the Weierstrass M-test, the series

$$\sum_{z\in\Lambda_k}\frac{\widehat{\varphi}(z)}{\prod_{j\neq k}\widehat{\mu_j}(z)}\left(\frac{1}{\frac{d}{d\zeta}\widehat{\mu_k}(z)}\right)\left(\frac{\widehat{\mu_k}(\zeta)}{\zeta - z}\right) \tag{30}$$

converge uniformly on compact subsets of $\mathbb{C}$, and therefore represent entire functions.

Now, for each $z \in \Lambda_k$ and $|\zeta - z| \geq \lambda > 0$ with $\lambda < 1/2r_k$, there exists a constant a_k

$$|\frac{\widehat{\mu_k}(\zeta)}{\zeta - z}| \leq \frac{1}{\lambda}|\widehat{\mu_k}(\zeta)| \leq \frac{a_k}{\lambda}e^{2\pi r_k|\Im\zeta|}$$

because $\widehat{\mu_k}(\zeta) = \frac{\sin(2\pi r_k\zeta)}{\pi\zeta}$. If $|\zeta - z| \leq \epsilon$, then

$$|\frac{\widehat{\mu_k}(\zeta)}{\zeta - z}| \leq \sup_{|\zeta - z|\leq\epsilon}|\frac{d}{d\zeta}\widehat{\mu_k}(\zeta)|\,.$$

Therefore, there exists $\kappa_k > 0$ such that

$$|\frac{\widehat{\mu_k}(\zeta)}{\zeta - z}| \leq \kappa_k e^{2\pi r_k|\Im\zeta|}$$

independently of $z \in \Lambda_k$.

From the estimates above, for $|z| < \rho_n$, we have that the series

$$\sum_{z\in\Lambda_k}\frac{|\widehat{\varphi}(z)|}{|\prod_{j\neq k}\widehat{\mu_j}(z)\frac{d}{d\zeta}\widehat{\mu_k}(z)|}$$

is bounded by some B. Therefore, for $|z| < \rho_n$,

$$\begin{aligned} & |\sum_{z\in\Lambda_k}\frac{\widehat{\varphi}(z)}{\prod_{j\neq k}\widehat{\mu_j}(z)\frac{d}{d\zeta}\widehat{\mu_k}(z)}\frac{\widehat{\mu_k}(\zeta)}{(\zeta - z)}| \\ \leq & \sum_{z\in\Lambda_k}\frac{|\widehat{\varphi}(z)|}{|\prod_{j\neq k}\widehat{\mu_j}(z)\frac{d}{d\zeta}\widehat{\mu_k}(z)|}\frac{|\widehat{\mu_k}(\zeta)|}{|\zeta - z|} \\ \leq & B\kappa_k e^{2\pi r_k|\Im\zeta|}, \end{aligned}$$

Thus, the series converges in $\widehat{\mathcal{E}'}$. This completes the proof of the lemma and the theorem. □

The conditions that we impose on our auxiliary function φ make it an approximate identity function. Its smoothness and the size of its support guarantee that the deconvolvers are compactly supported. We can discuss more general conditions by introducing the following. Let $k \in \mathbb{N}$, and let $\alpha = k + \beta$, where $0 \leq \beta < 1$. We define the *Hölder space* C^α *of exponent* α as

$$\left\{ f \in C^k : \text{There exists } M > 0 \text{ such that } \ |f^{(k)}(t) - f^{(k)}(\tau)| \leq M|t-\tau|^\beta \right\}.$$

We also define the related class D^α as

$$\left\{ f \colon \widehat{f}(\omega)\,(1+|\omega|)^\alpha \in L^\infty \right\},$$

for $\omega \in \widehat{\mathbb{R}}$. We have that

$$D^{\alpha+1} \subset C^\alpha.$$

If f has compact support and $f \in C^\alpha$, then $f \in D^\alpha$. For the compactly supported function φ, we have that if $\varphi \in C^{3+\eta}$, $\eta > 0$,

$$|\widehat{\varphi}(z)| \leq \frac{d}{(1+|z|)^{3+\eta}}$$

for $z \in \Gamma_n$.

We now give explicit formulae for the deconvolvers without using inverse transforms.

Lemma 3 *If μ is a compactly supported function such that $\widehat{\mu}(\beta) = 0$, then*

$$\frac{\widehat{\mu}(\zeta)}{(\zeta-\beta)} = \widehat{T_\beta}(\zeta),$$

where

$$\langle T_\beta; \varphi \rangle = -2\pi i \int_{\mathbb{R}} \left(\int_0^x \varphi(t) e^{2\pi i \beta (t-x)} dt \right) \mu(t)\, dx.$$

Proof : Since $T_\beta \in \mathcal{E}'$,

$$\begin{aligned}
\widehat{T_\beta} &= \langle T_\beta; e^{-2\pi i t\zeta} \rangle \\
&= -2\pi i \int_{-\infty}^{\infty} \left(\int_0^x e^{-2\pi i t\zeta} e^{2\pi i\beta(t-x)} dt \right) \mu(t)\, dx \\
&= -2\pi i \int_{-\infty}^{\infty} \left(\int_0^x e^{-2\pi i t(\zeta-\beta)} dt \right) \mu(t) e^{-2\pi i x\beta}\, dx \\
&= \int_{-\infty}^{\infty} \left(-2\pi i \frac{e^{-2\pi i x\zeta} e^{2\pi i x\beta} - 1}{-2\pi i\,(\zeta-\beta)} \right) \mu(t) e^{-2\pi i x\beta}\, dx \\
&= \int_{-\infty}^{\infty} \frac{\mu(t) e^{-2\pi i x\zeta}}{(\zeta-\beta)}\, dx - \int_{-\infty}^{\infty} \frac{\mu(t) e^{-2\pi i x\beta}}{(\zeta-\beta)}\, dx \\
&= \frac{\widehat{\mu}(\zeta) - \widehat{\mu}(\beta)}{(\zeta-\beta)} = \frac{\widehat{\mu}(\zeta)}{(\zeta-\beta)},
\end{aligned}$$

since $\widehat{\mu}(\beta) = 0$. □

Lemma 4 *If $\mu_k(t) = \chi_{[-r_k,r_k]}(t)$ and $z \in \Lambda_k$, then*

$$\frac{\widehat{\mu_k}(\zeta)}{(\zeta - z)} = \widehat{T_{k,z}}(\zeta),$$

where

$$T_{k,z}(t) = \frac{1}{z}\left(e^{2\pi i z(t+r_k)} - 1\right)\chi_{[-r_k,r_k]}(t).$$

Proof :

$$\begin{aligned}
\langle T_{k,z};\varphi\rangle &= -2\pi i\int_{-\infty}^{\infty}\left(\int_0^x \varphi(t)e^{2\pi i z(t-x)}\,dt\right)\chi_{[-r_k,r_k]}(x)\,dx \\
&= -2\pi i\int_{-r_k}^{r_k}\left(\int_0^x \varphi(t)e^{2\pi i z(t-x)}\,dt\right)dx \\
&= 2\pi i\int_{-r_k}^{0}\left(\varphi(t)\int_{-r_k}^{t} e^{2\pi i z(t-x)}\,dx\right)dt \\
&\quad + -2\pi i\int_0^{r_k}\left(\varphi(t)\int_t^{r_k} e^{2\pi i z(t-x)}\,dx\right)dt \\
&= \left\langle \frac{1}{z}\left(e^{2\pi i z(t+r_k)} - 1\right)\chi_{[-r_k,r_k]}(t);\varphi\right\rangle .
\end{aligned}$$

□

These last two lemmas give us explicit formulae for the $\nu_{k,\varphi}$ in the time domain. We add that the estimates given in the proof of the previous theorem give us that φ only needs to be a compactly supported $C^{3+\eta}$ function, because this is sufficient to guarantee that the series representations for the $\nu_{k,\varphi}$ converge to compactly supported functions.

Theorem 9 *Let $f \in C^\infty \cap L^2(\mathbb{R})$, and for $\eta > 0$ let φ be a function in the Hölder space $C^{3+\eta}$ with support in $[-A, A]$ such that $0 < A < R = \sum r_i$, $\varphi \geq 0$, and $\int_{\mathbb{R}} \varphi(t)\,dt = 1$. Let σ be any permutation of $\{1, 2, \ldots, m\}$ such that $\sigma(i) \neq i$. Given $\{r_i\}_{i=1}^m$ such that (r_i/r_j) is poorly approximated by rationals for $i \neq j$ and convolvers*

$$\left\{\chi_{[-r_i,r_i]}\right\}, i = 1, \ldots, m\,,$$

the deconvolvers $\nu_{k,\varphi}$ such that

$$f * \varphi = \sum_{i=1}^{m}(f * \mu_i) * \nu_{i,\varphi}$$

are given by the formulae

$$\nu_{k,\varphi}(t) = \alpha_{\sigma(k),\varphi} * K(t) \tag{31}$$

where $K(t)$ is the convolution of μ_j for $j \neq k, \sigma(k)$, and

$$\alpha_{k,\varphi}(t) = \sum_{z\in\Lambda_k} \frac{\widehat{\varphi}(z)}{\prod_{j\neq k}\widehat{\mu_j}(z)} \left(\frac{1}{\frac{d}{d\zeta}\widehat{\mu_k}(z)}\right) \left(\frac{1}{z}\left(e^{2\pi i z(t+r_k)} - 1\right)\chi_{[-r_k,r_k]}(t)\right). \tag{32}$$

The function $f * \varphi$ is an arbitrarily close approximation of f which converges to f in the sense of distributions as supp$(\varphi) \longrightarrow \{0\}$.

To finish this section, we need to show that the deconvolution formulae given in Theorems 8 and 9 work for general L^2 functions. We know that $C^\infty \cap L^2(\mathbb{R})$ is dense in L^2. We also note that the functions $\nu_{k,\varphi}$ are compactly supported and continuous. Therefore, since $f * \mu_k \in L^2$, $f * \mu_k * \nu_{k,\varphi} \in L^2$ with

$$\|f * \mu_k * \nu_{k,\varphi}\|_2 \leq \|f * \mu_k\|_2 \|\nu_{k,\varphi}\|_1.$$

Similarly,

$$\|f * \varphi\|_2 \leq \|f\|_2 \|\varphi\|_1.$$

Define the operator

$$\mathcal{D}\colon L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

by

$$\mathcal{D}(f) = f * \varphi - \left(\sum_{i=1}^{k} f * \mu_i * \nu_{i,\varphi}\right).$$

The operator $\mathcal{D}$ is clearly linear. The estimates above give us that $\mathcal{D}$ is bounded, and therefore continuous. Since $\mathcal{D} = 0$ on $C^\infty \cap L^2(\mathbb{R})$ and is continuous, $\mathcal{D} = 0$ on $L^2(\mathbb{R})$. Thus, the deconvolution procedure works for general L^2 functions.

Remark 3 (a) A several variable modification of this construction may be found in [13]. This construction requires a much stronger form of the strongly coprime condition, as this is needed to solve the Bezout equation in several variables. The construction of the transformed deconvolvers is achieved by solving the analytic Bezout equation using the known values on the zero sets of the $\widehat{\mu}_i$. These may be thought of as interpolation formulae for several complex variables as they reconstruct analytic functions based on distinct values at a discrete set of isolated points. The ν_i are constructed simultaneously by taking the inverse Fourier-Laplace transform of the formulae.

(b) The deconvolvers are not unique. For example, given two deconvolvers ν_1 and ν_2, $\lambda \in \mathbb{R}$, and a compactly supported function η, the pair

$$\tilde{\nu}_1 = \nu_1 + \lambda\eta * \mu_2\ ,\ \tilde{\nu}_2 = \nu_2 - \lambda\eta * \mu_1$$

is also a set of deconvolvers. The non-uniqueness of the deconvolvers allows one to develop deconvolvers that are optimal with respect to a given condition. In [8] and [20], deconvolvers that optimize the signal to noise ratio of the signal relative to white Gaussian noise were constructed. Other types of noise could be dealt with in similar fashion.

3 Sampling on Non-Commensurate Lattices

The second step in solving the Multichannel Deconvolution Problem (MDP) involves solving an interpolation problem, reconstructing functions (the deconvolvers) in a space of restricted growth ($\widehat{\mathcal{E}'}$) from discrete data (their values on the zero sets of the convolvers). This gives solutions to the Bezout equation. This development utilizes the zero sets of the $\widehat{\mu_i}$ as different sampling rates. This connection between sampling and multichannel deconvolution naturally leads to sampling schemes on properly chosen non-commensurate lattices. We now consider our second problem: Let $\{r_i\}_{i=1}^m$ be a set such that r_i/r_j is *poorly approximated by rationals* for $i \neq j$, and let $R = \sum_{i=1}^m r_i$. Let

$$\Lambda = \bigcup_{i=1}^{m} \left\{ \frac{\pm k}{2r_i} \right\}_{k \in \mathbb{N}} \bigcup \{0\}$$

be a sampling grid, made up of a union of sampling grids with non-commensurate generators $\{r_i\}$. Given a R bandlimited function f, we can reconstruct f from samples of f taken on Λ. We refer to this as the *Multi-Rate Sampling Problem* or MRSP. We begin this section by using the Cauchy Residue Calculus to derive the Whittaker-Kotel'nikov-Shannon (W-K-S) Sampling Theorem.

3.1 W-K-S Sampling via Complex Interpolation

Theorem 10 (W-K-S Sampling Theorem) *Let f be a function of finite energy on $\mathbb{R}$ ($f \in L^2(\mathbb{R})$) with Fourier transform $\widehat{f}(\omega) = 0$ for all $|\omega| \geq \Omega$, i.e., $f(t)$ is Ω-band-limited.*

a.) *If $T \leq 1/2\Omega$, then for all $t \in \mathbb{R}$,*

$$f(t) = T \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin(\frac{\pi}{T}(t - nT))}{\pi(t - nT)}.$$

b.) *If $T \leq 1/2\Omega$ and $f(nT) = 0$ for all $n \in \mathbb{Z}$, then $f \equiv 0$.*

Let f be an Ω-band-limited function of finite energy on $\mathbb{R}$. (Also works for $f \in L^p(\mathbb{R})$, $1 < p < \infty$.) To simplify notation, we assume

$$\Omega \leq \frac{1}{2}\,.$$

Proof: Given f which satisfies the hypothesis of the theorem, $f(t)$ is real analytic and has an analytic continuation $f(z)$ to $\mathbb{C}$. Moreover, $f(z)$ satisfies the Paley-Wiener growth bound for Ω, i.e., there exists $K = K(n)$ such that for all $n \in \mathbb{N}$

$$|f(z)| \leq K(n)(1+|z|)^{-n} e^{2\pi\Omega|\Im z|}\,.$$

Let $T = 1$. Then $T \leq \frac{1}{2\Omega}$. Also let

$$\widehat{\mu}(\zeta) = \sin(\pi\zeta)\,.$$

This has zeros $\mathcal{Z} = k$, $k \in \mathbb{Z}$. To avoid these zeros, let C_m be a circle centered at the origin with radius $(m + \frac{1}{2})$, for $m \in \mathbb{N}$. We now apply the Jacobi Interpolation Formula.

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C_m} \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i}\oint_{C_m} \frac{f(\zeta)}{\zeta - z}\cdot\frac{\widehat{\mu}(\zeta)-\widehat{\mu}(z)}{\widehat{\mu}(\zeta)} d\zeta + \frac{1}{2\pi i}\oint_{C_m} \frac{f(\zeta)}{\zeta - z}\cdot\frac{\widehat{\mu}(z)}{\widehat{\mu}(\zeta)} d\zeta \\ &= \qquad\qquad (I) \qquad\qquad + \qquad\qquad (II) \end{aligned}$$

We will show below that $(II) \longrightarrow 0$ as $m \longrightarrow \infty$. A careful estimation of this second integral provides a bound for the truncation error in our sampling formula.

Assuming $(II) \longrightarrow 0$, we are left with (I).

$$\begin{aligned} f(z) &\approx \frac{1}{2\pi i}\oint_{C_m} \frac{f(\zeta)}{\zeta - z}\cdot\frac{\widehat{\mu}(\zeta)-\widehat{\mu}(z)}{\widehat{\mu}(\zeta)} d\zeta \\ &= \frac{\widehat{\mu}(z)}{2\pi i}\oint_{C_m} \frac{f(\zeta)}{\zeta - z}\cdot\left[\frac{1}{\widehat{\mu}(z)} - \frac{1}{\widehat{\mu}(\zeta)}\right] d\zeta \end{aligned}$$

At this point, we need to note a couple items before proceeding. First

$$\sin\pi\zeta = \lim_{N\to\infty} s_N(\zeta) = \lim_{N\to\infty} \pi\zeta\prod_{j=1}^{N}(1-\frac{\zeta^2}{j^2})$$

is the Weierstrass product representation of a sine function. Second, we have the following Mittag-Leffler partial fractions decomposition

$$\frac{1}{(\zeta - z)s_N(\zeta)} = \frac{1}{(\zeta - z)s_N(z)} + \sum_{|n|\leq N}\frac{1}{(\zeta - n)(n - z)s'_N(n)}$$

This can be shown using the fact that

$$\frac{1}{p(\zeta)} = \sum_{j=1}^{s}\frac{1}{(\zeta - r_j)p'(r_j)}$$

where $p(\zeta)$ is a polynomial with only simple zeros at $\{r_j\}_{j=1}^{s}$.

Let $R_m = (II)$ for ζ inside of C_m. Then, if $m < N$,

$$\begin{aligned}
f(z) &= \frac{\widehat{\mu}(z)}{2\pi i}\oint_{C_m}\frac{f(\zeta)}{\zeta - z}\cdot\left[\frac{1}{\widehat{\mu}(z)} - \frac{1}{\widehat{\mu}(\zeta)}\right]d\zeta \;+\; R_m \quad (1.)\\
&= \lim_{N\to\infty}\frac{s_N(z)}{2\pi i}\oint_{C_m}\frac{f(\zeta)}{\zeta - z}\cdot\left[\frac{1}{s_N(z)} - \frac{1}{s_N(\zeta)}\right]d\zeta \;+\; R_m \quad (2.)\\
&= \lim_{N\to\infty}\frac{s_N(z)}{2\pi i}\oint_{C_m}\cdot\sum_{|n|\le N}\frac{f(\zeta)}{(\zeta - n)(n - z)s_N'(n)}d\zeta \;+\; R_m \quad (3.)\\
&= \lim_{N\to\infty}\frac{s_N(z)}{2\pi i}\sum_{|n|\le N}\cdot\oint_{C_m}\frac{f(\zeta)}{(\zeta - n)(n - z)s_N'(n)}d\zeta \;+\; R_m \quad (4.)\\
&= \lim_{N\to\infty} s_N(z)\sum_{|n|\le m}\frac{f(n)}{(z - n)s_N'(n)} \;+\; R_m \quad (5.)\\
&= \sin\pi z\sum_{|n|\le m} f(n)\cdot\frac{(-1)^n}{\pi(z - n)} \;+\; R_m \quad (6.)\\
&= \sum_{|n|\le m} f(n)\,\mathrm{sinc}(z - n) \;+\; R_m \quad (7.)
\end{aligned}$$

where (1.) is Jacobi interpolation, (2.) is the Weierstrass product, (3.) is the Mittag-Leffler decomposition, (4.) is the switch between limit and integration permitted by uniform convergence, (5.) is the Cauchy Residue Calculus, (6.) is the Weierstrass product (and trigonometric evaluation), and (7.) is the definition of the sinc function. Since $R_m \longrightarrow 0$ as $m \longrightarrow \infty$, we get the sampling formula.

To finish, we have to estimate the second integral above.

Lemma 5

$$\left|\frac{1}{2\pi i}\oint_{C_m}\frac{f(\zeta)}{\zeta - z}\cdot\frac{\widehat{\mu}(z)}{\widehat{\mu}(\zeta)}d\zeta\right| \longrightarrow 0 \;\; as \; m \longrightarrow \infty\,.$$

Proof: We have that C_m is a circle centered at the origin with radius $(m + \frac{1}{2})$ for $m \in \mathbb{N}$. Thus,

$$\left|\frac{1}{2\pi i}\oint_{C_m}\frac{f(\zeta)}{\zeta - z}\cdot\frac{\widehat{\mu}(z)}{\widehat{\mu}(\zeta)}d\zeta\right| = \left|\frac{1}{2\pi i}\int_0^{2\pi}\frac{\sin(\pi z)}{\zeta - z}\left[\frac{f((m + 1/2)e^{i\theta})}{\sin(\pi(m + 1/2)e^{i\theta})}i((m + 1/2)e^{i\theta})\right]d\theta\right|.$$

Now let z be contained in the closed ball of R centered at the origin, with $R < m + 1/2$. Since $\zeta \in C_m$

$$|\zeta - z| \ge (m + 1/2) - R\,.$$

Since $f(z)$ satisfies the Paley-Wiener growth bound for Ω, there exists $K = K(n)$ such that for all $n \in \mathbb{N}$

$$|f(z)| \leq K(n)(1+|z|)^{-n} e^{2\pi\Omega|\Im z|} .$$

For $z = x + iy$,

$$|\sinh(y)| \leq |\sin(z)| \leq |\cosh(y)| .$$

Choose δ sufficiently small so that for $|\theta| < \delta$,

$$|\sin(\pi(m+1/2)\cos(\theta))| \geq \frac{1}{2} .$$

Then, for $|\theta| < \delta$,

$$|\sin(\pi(m+1/2)\cos(\theta))| \geq \frac{1}{2}\exp(\pi(m+1/2)|\sin(\theta)|) .$$

Thus,

$$\begin{aligned}
(II) &= \left| \frac{1}{2\pi i} \oint_{C_m} \frac{f(\zeta)}{\zeta - z} \cdot \frac{\widehat{\mu}(z)}{\widehat{\mu}(\zeta)} d\zeta \right| \\
&= \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{\sin(\pi z)}{\zeta - z} \left[\frac{f((m+1/2)e^{i\theta})}{\sin(\pi(m+1/2)e^{i\theta})} i((m+1/2)e^{i\theta}) \right] d\theta \right| \\
&\leq |\cosh(\pi y)| \frac{K(n)}{[3/2+m]^n} \exp(\pi(m+1/2)(\Omega-1)) .
\end{aligned}$$

For $0 < \Omega < 1$,

$$\left| \frac{1}{2\pi i} \oint_{C_m} \frac{f(\zeta)}{\zeta - z} \cdot \frac{\widehat{\mu}(z)}{\widehat{\mu}(\zeta)} d\zeta \right| = O\left(e^{m(\Omega-1)} \right)$$

which $\longrightarrow 0$ exponentially as $m \longrightarrow \infty$. For $\Omega = 1$,

$$\left| \frac{1}{2\pi i} \oint_{C_m} \frac{f(\zeta)}{\zeta - z} \cdot \frac{\widehat{\mu}(z)}{\widehat{\mu}(\zeta)} d\zeta \right| = O\left(\frac{1}{[1+m]^n} \right)$$

which $\longrightarrow 0$ with polynomial decay as $m \longrightarrow \infty$. This completes the proof of the lemma and the proof of the theorem. □

3.2 Multi-Rate Sampling

Let $t \in \mathbb{R}$, and let α be an irrational that is poorly approximated by rationals, e.g., $\sqrt{2}$, or $(1+\sqrt{5})/2$. Let $\mu_1(t) = \chi_{[-1,1]}(t)$, $\mu_2(t) = \chi_{[-\alpha,\alpha]}(t)$ model the impulse response of the channels of a two-channel system. Then

$$\widehat{\mu_1}(\zeta) = \frac{\sin(2\pi\zeta)}{\pi\zeta} , \ \widehat{\mu_2}(\zeta) = \frac{\sin(2\pi\alpha\zeta)}{\pi\zeta} .$$

Since α is poorly approximated by rationals, $\{\mu_i\}$ is strongly coprime. Now let

$$\Lambda_1 = \left\{ \frac{\pm n}{2} \right\} \, , \; \Lambda_2 = \left\{ \frac{\pm m}{2\alpha} \right\} ,$$

for $n, m \in \mathbb{N}$, and let

$$\Lambda = \Lambda_1 \cup \Lambda_2 \, .$$

Note that the information contained in the original signal is reconstructed by creating deconvolvers defined initially on $\Lambda \cup \{0\}$. Order the elements of Λ, denoting this as $\Lambda = \{\lambda_k\}$. Walnut [60] has shown the following.

Theorem 11 *Let α be an irrational that is poorly approximated by rationals, and let f be a $(1+\alpha)$-band-limited function. Then f is uniquely determined by*

$$\{f(\lambda_k)\} \cup \{f(0), f'(0)\} \, .$$

Note, for a $(1+\alpha)$-band-limited function, the Nyquist rate is $1/(2(1+\alpha))$. However, our individual sampling rates are $1/2$ and $1/(2\alpha)$. Both these rates are below Nyquist. The reconstruction of f from this lattice is achieved by using complex interpolation theory. These techniques go back to basic Lagrange interpolation, and were developed for entire functions by various mathematicians, most notably B. Ya. Levin [44]. Let

$$G(z) = \sin(2\pi z) \sin(2\pi\alpha z) \, . \tag{33}$$

Then, $G(z)$ is an entire function, which is almost periodic on $\mathbb{R}$, and has simple zeros on Λ, and a double zero at $\{0\}$.

Proposition 1 *Let $\lambda_k \in \Lambda$ and let*

$$H_j(z) = \frac{G(z)}{G'(\lambda_j)(z - \lambda_j)} \, . \tag{34}$$

Then

$$H_j(\lambda_k) = \delta_{jk} \, . \tag{35}$$

Proof : Let $\lambda_j = \frac{\pm n}{2}$, $\lambda_k = \frac{\pm m}{2\alpha}$. Using the Taylor expansion in a neighborhood of λ_j, we have that $\sin(2\pi z) = 2\pi(-1)^n(z - \lambda_j) + O((z - \lambda_j)^3)$. Similarly, in a neighborhood of λ_k, we have that $\sin(2\pi\alpha z) = 2\pi\alpha(-1)^m(z - \lambda_k) + O((z - \lambda_k)^3)$. Therefore, in a neighborhood of λ_j

$$H_j(z) = \frac{G(z)}{G'(\lambda_j)(z - \lambda_j)} = \frac{2\pi(-1)^n(z - \lambda_j)\sin(2\pi\alpha\lambda_j)}{2\pi(-1)^n(z - \lambda_j)\sin(2\pi\alpha\lambda_j)} + O((z - \lambda_j)^2) = 1 + O((z - \lambda_j)^2) \, .$$

Similarly, in a neighborhood of λ_k

$$H_k(z) = \frac{G(z)}{G'(\lambda_k)(z-\lambda_k)} = \frac{2\pi\alpha(-1)^m(z-\lambda_k)\sin(2\pi\lambda_k)}{2\pi\alpha(-1)^m(z-\lambda_k)\sin(2\pi\lambda_k)} + O((z-\lambda_k)^2) = 1 + O((z-\lambda_k)^2)\,.$$

□

At $z = 0$, we have to construct interpolating functions K_1, K_2 so that

$$K_1(0) = 1\,,\ K_1'(0) = 0\,,$$

$$K_2(0) = 0\,,\ K_2'(0) = 1\,.$$

Using the Taylor expansion of G, we derive the following.

Proposition 2

$$K_1(z) = \frac{G(z)}{(G''(0)/2!)(z^2)} = \frac{G(z)}{(4\pi^2\alpha)(z^2)}\,, \tag{36}$$

$$K_2(z) = \frac{G(z)}{(G''(0)/2!)(z)} = \frac{G(z)}{(4\pi^2\alpha)(z)}\,. \tag{37}$$

Proof : Using the Taylor expansion

$$\begin{aligned} G(t) &= \sin(2\pi t)\cdot\sin(2\pi\alpha t) \\ &= [2\pi t - \frac{(2\pi t)^3}{3!} + O(t^5)]\cdot[2\pi\alpha t - \frac{(2\pi\alpha t)^3}{3!} + O(t^5)] \\ &= 4\pi^2\alpha t^2 - \frac{32\pi^4\alpha}{3!}t^4 + O(t^6) \end{aligned}$$

Thus, we have expressions for $\frac{G(t)}{t^i}$, $i = 1, 2$:

$$\begin{aligned} \frac{G(t)}{t} &= 4\pi^2\alpha t - \frac{32\pi^4\alpha}{3!}t^3 + O(t^5) \\ \frac{G(t)}{t^2} &= 4\pi^2\alpha - \frac{32\pi^4\alpha}{3!}t^2 + O(t^4) \end{aligned}$$

with derivatives:

$$\begin{aligned} \frac{d}{dt}\frac{G(t)}{t} &= 4\pi^2\alpha - \frac{96\pi^4\alpha}{3!}t^2 + O(t^4) \\ \frac{d}{dt}\frac{G(t)}{t^2} &= -\frac{64\pi^4\alpha}{3!}t + O(t^3) \end{aligned}$$

Computing limits of the functions and their derivatives at $t = 0$ gives

$$K_1(t) = \frac{G(t)}{\frac{G''(0)}{2!}t^2}\ ,\ K_2(t) = \frac{G(t)}{\frac{G''(0)}{2!}t}$$

where $\frac{G''(0)}{2!} = 4\pi^2\alpha$. □

Combining these two propositions gives us the reconstruction formula.

Theorem 12 *Let α be an irrational that is poorly approximated by rationals, and let f be a $(1+\alpha)$-band-limited function. Let*

$$\Lambda_1 = \left\{ \frac{\pm k}{2} \right\} \ , \ \Lambda_2 = \left\{ \frac{\pm k}{2\alpha} \right\},$$

for $k \in \mathbb{N}$, and let

$$\Lambda_1 \cup \Lambda_2 = \Lambda = \{\lambda_k\}\,.$$

Then f is uniquely determined by

$$\{f(\lambda_k)\} \cup \{f(0), f'(0)\}\,.$$

Moreover, f can be approximated from its values on $\Lambda \cup \{0\}$ by the formula

$$\begin{aligned} f(t) \approx \sum_{\lambda_k \in \Lambda} f(\lambda_k) \frac{G(t)}{G'(\lambda_k)(t-\lambda_k)} \\ + f(0)\frac{G(t)}{(4\pi^2\alpha)(t^2)} \ + \ f'(0)\frac{G(t)}{(4\pi^2\alpha)(t)}\,, \end{aligned} \tag{38}$$

where

$$G(t) = \sin(2\pi t) \cdot \sin(2\alpha\pi t)\,. \tag{39}$$

Convergence is uniform convergence on compact subsets, and is shown in a manner similar to the results for deconvolution. The proof is lengthy, and will be shown elsewhere. We need to point out two items. First, the sampling grid is rigid. Perturbation of the grid results in a loss of information. Second, that because sampling points in Λ can get arbitrarily close together ($\inf\{|\lambda_m - \lambda_n|\} = 0$, and so $\inf\{\prod_{k\neq j} |\sin(2\pi r_k \lambda)| : \lambda \in \Lambda_j\} = 0$), the set of interpolating functions can not form a Riesz basis and the interpolating formula can not converge in norm [45, 46]. In fact, the interpolating functions do form a Bessel sequence, but do not form a frame and therefore do not form a Riesz basis. The problems occur at points where the sampling points get close together. The interpolating function follows the original function along exactly, except for a very subtle "ripple" at those points where the sampling points get close together. The following stabilizes the construction. Let δ be given, $0 < \delta < \frac{1}{4\alpha}$. Let

$$\Lambda_\delta = \{\lambda \in \Lambda : \mathrm{dist}(\lambda, \Lambda \setminus \{\lambda\}) < \delta\}\,.$$

Elements in Λ_δ occur in pairs, with each pair containing one element from $\Lambda_1 = \left\{\frac{\pm k}{2}\right\}$ and the other from $\Lambda_2 = \left\{\frac{\pm k}{2\alpha}\right\}$ for $k \in \mathbb{N}$. Let $\{\lambda_k\} = \Lambda = \Lambda_1 \cup \Lambda_2 = \Lambda_\delta \cup \Lambda_\delta{}'$. For

$$G(t) = \sin(2\pi t) \cdot \sin(2\pi\alpha t), \text{ let } H_{\lambda_k} = \frac{G(t)}{G'(\lambda_k)(t-\lambda_k)}\,.$$

We can show that

$$\text{Span}\left\{\frac{G(t)}{G'(\lambda_k)(t-\lambda_k)}\right\}$$

is dense in $L^2_{(1+\alpha)}$ and that

$$\left\{\frac{G(t)}{G'(\lambda_k)(t-\lambda_k)}\right\}_{\lambda_k\in\Lambda_\delta{}'}$$

is a Bessel sequence. Also, we can show that

$$\frac{G(t)}{4\pi\alpha t}\,,\ \frac{G(t)}{4\pi\alpha t^2}\in \text{Closure}\left(\text{Span}\left\{\frac{G(t)}{G'(\lambda_k)(t-\lambda_k)}\right\}\right)\,.$$

The points in Λ_δ can be treated as double points. For $\{\lambda_1,\lambda_2\}\subset\Lambda_\delta$,

$$\begin{aligned}
&[f(\lambda_1)H_{\lambda_1}(t)+f(\lambda_2)H_{\lambda_2}(t)]\\
&= f(\lambda_1)H_{\lambda_1}(t)+f'(\xi_{\lambda_1})\left[\frac{G(t)}{\left(\frac{G'(\lambda_1)-G'(\lambda_2)}{2(\lambda_1-\lambda_2)}\right)(t-\lambda_1)(t-\lambda_2)}\right]\\
&+ \mathcal{R}(\lambda_1-\lambda_2)^2\,,
\end{aligned}$$

where $\xi_{\lambda_1}\longrightarrow\lambda_1$ and $\mathcal{R}(\lambda_1-\lambda_2)^2\longrightarrow 0$ as $\delta\longrightarrow 0$. Finally, we can show that

$$\left\{\frac{G(t)}{(G''(\lambda_k)/2!)(t-\lambda_k)^2}\right\}_{\lambda_k\in\Lambda_\delta}$$

is a Bessel sequence.

The result generalizes. We can create sampling sets on ℓ lattices using a set of numbers $\{r_i\}_{i=1}^{\ell}$ such that (r_i/r_j) is poorly approximated by rationals for $i\neq j$. Again, if $\{p_i\}_{i=1}^{\ell-1}$ is a set of primes,

$$\{1,\sqrt{p_1},\sqrt{p_1p_2},\ldots,\sqrt{p_1p_2\cdots p_{\ell-1}}\}$$

is a set of numbers whose ratios are poorly approximated by rationals. Let $\Lambda_k=\left\{\frac{\pm n}{2r_k}\right\}$ for $n\in\mathbb{N}$, and let

$$\bigcup_{k=1}^{\ell}\Lambda_k=\Lambda=\{\lambda_k\}\,. \tag{40}$$

We reconstruct on $\Lambda\cup\{0\}$, letting

$$G(z)=\prod_{k=1}^{\ell}\sin(2\pi r_k z) \tag{41}$$

and letting

$$H_m(z)=\frac{G(z)}{G'(\lambda_m)(z-\lambda_m)}\,. \tag{42}$$

Then

$$H_m(\lambda_n) = \delta_{mn}\,. \tag{43}$$

The interpolating functions at the origin are a linear combination of

$$g_j = \frac{G(z)}{(z^j)}\,,\; j = 1, \ldots, \ell\,, \tag{44}$$

chosen so that

$$K_k^{(j-1)}(0) = \delta_{kj}\,,\; k, j = 1, \ldots, \ell\,. \tag{45}$$

As before, using Taylor series, we get

$$\begin{aligned} H_k^{(j)}(t) &= 2^n\pi^n \frac{(n-k)!}{(n-k-j)!}\sqrt{p_1 p_2 \cdots p_n}\, t^{n-k-j} \\ &\quad -\frac{2^{n+2}\pi^{n+2}}{3!}\frac{(n-k+2)!}{(n-k-j+2)!}\sqrt{p_1 p_2 \cdots p_n} \\ &\quad (1 + \sum_{i=1}^{n-1} p_i) t^{n-k-j+2} \\ &\quad + O(t^{n-k-j+4})\,. \end{aligned} \tag{46}$$

We want

$$K_k(t) = \sum_{i=1}^{k} c_i H_i(t) \text{ so that } K_k^{(j-1)}(0) = \delta_{kj}\,,\; k, j = 1, \ldots, \ell\,. \tag{47}$$

Solving these relationships gives

$$\begin{aligned} K_k(t) &= \sum_{i=1}^{k} c_i H_i(t) \\ &= \frac{\sum_{m=0}^{\lceil (n-k)/2 \rceil} (\frac{2^2\pi^2}{3!}(1 + \sum_{i=1}^{n-1} p_i))^{2m} t^{2m} G(t)}{\frac{G^{(n)}(0)}{(n)!} \cdot (k-1)! \cdot t^{n-k+1}}\,. \end{aligned} \tag{48}$$

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MULTISCALE APPROXIMATION FOR EIGENVALUE PROBLEMS OF FREDHOLM INTEGRAL EQUATIONS

En-Bing Lin

Abstract.

We introduce multiscaling approximation for eigenvalue problems of Fredholm integral equation of the second kind and prove an eigenvalue approximation theorem.

1. Introduction

We consider the following homogeneous Fredholm equation of the second kind.

$$Ku = \lambda u \tag{1.1}$$

where $Ku(x) = \int_0^1 \ k(x,y)\ u(y)\ dy$ with a smooth kernel k and some unknown function u on [0, 1].

Several methods are known for the approximate solutions of the above equation [LS]. Some of these methods base on the partitions of the domain or use some linearly independent functions such as iterated-collocation method and Galerkin method. Most of these methods solve one dimensional problems. Here we consider a different approach, namely, scaling function interpolation methods which can be used to solve higher dimensional problems as well.

Multiresolution based interpolation methods are relatively new in approximation theory and become useful and powerful in several areas of mathematical analysis and applications. In this paper, we prove an eigenvalue approximation theorem based on multiresolution analysis and propose an open problem along this line. The outline of this paper is as follows. In this section, we recall scaling interpolation method in scalar case, we then recall multiscaling interpolation method in section 2 and 3 followed by some practical and useful formulas in section 4. The main result on eigenvalue approximation of (1.1) is presented in section 5.

To review several approximation theorems obtained by using scaling function interpolation, we first recall the setting of multiresolution analysis.

Suppose $\phi(x)$ and $\psi(x)$ are the scaling function and the corresponding wavelet respectively with finite support [0, l] where l is a positive number. It is well known that $\phi(x)$ and $\psi(x)$ satisfy the following dilation equation and wavelet equation, respectively:

$$\phi(x) = \sqrt{2}\ \sum_{s=0}^{l}\ h_s \phi(2x - s)$$

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and

$$\psi(x) = \sqrt{2} \sum_{s=0}^{l} g_s \phi(2x - s) \ ,$$

where the $h'_s s$ and $g'_s s$ are constants called low pass and high pass filter coefficients respectively.

We will use the following standard notations:

$$\phi_k^j(x) = 2^{\frac{j}{2}} \phi(2^j x - k),$$

and

$$\psi_k^j(x) = 2^{\frac{j}{2}} \phi(2^j x - k).$$

Consider the subspace V_j of L^2 defined by :

$$V_j = Span\{\phi_k^j, k \in Z\},$$

and the subspace W_j of L^2 defined by :

$$W_j = Span\{\psi_k^j, k \in Z\},$$

the subspaces $V'_j s, -\infty < j < \infty$, form a multiresolution of L^2 with the subspace W_j being the difference between V_j and V_{j-1}. In fact, the L^2 space has an orthonormal decomposition as :

$$L^2 = V_j \bigoplus \sum_{J=j}^{\infty} W_J.$$

The projection of a L^2 function $f(x)$ onto the subspace V_j is defined by:

$$f_j(x) = \sum_k \alpha_{j,k} \phi_k^j(x),$$

where

$$\alpha_{j,k} = \int f(x) \phi_k^j(x) dx.$$

Similarly, we can project $f(x)$ onto W_j by:

$$w_j(x) = \sum_k \beta_{j,k} \psi_k^j(x),$$

where

$$\beta_{j,k} = \int f(x) \psi_k^j(x) dx.$$

Therefore, the function $f(x)$ can be decomposed by:

$$f(x) = f_j(x) + \sum_{i=j}^{\infty} w_i(x).$$

The projection $f_j(x)$ is called the *linear approximation* of the function $f(x)$ in the subspace V_j [WS]. The multiresolution analysis of $L^2(R^2)$ can be defined in a similar fashion.

In [LZ], E. B. Lin and X. Zhou gave the following wavelet approximation theorem. In what follows, we assume ϕ, ψ are sufficiently smooth and satisfy the orthonormal multiresolution analysis with compact support and ψ has N vanishing moments. Let $\{M_l\}$ denote the moments of the scaling function ϕ, i.e.

$$M_l := \int x^l \phi(x) dx, \quad l = 1, 2, \cdots .$$

In particular,

$$M_1 = c := \frac{1}{2} \sum_k k a_k,$$

where $a_k = \sqrt{2} h_k$ are coefficients of the dilation equation.

Theorem 1.1. *Assume the function $f \in C^k(\bar{\Omega})$, where Ω is a bounded open set in $R^2, k \geq N \geq 2$. Let, for $j \in Z$,*

$$f^j(x, y) := \frac{1}{2^j} \sum_{p,q \in \wedge} f(\frac{p+c}{2^j}, \frac{q+c}{2^j}) \phi_p^j(x) \phi_q^j(y), \quad (x, y) \in \Omega, \tag{1.2}$$

where the index set $\wedge = \{(p, q) | (supp(\phi_p^j) \otimes supp(\phi_q^j)) \cap \Omega \neq \phi\}$, In addition the moments M_i satisfy

$$M_i = c^i, \quad i = 1, 2, \cdots, N-1.$$

i.e.

$$\int x^i \phi(x) dx = (\int x \phi(x) dx)^i, i = 1, 2, \cdots, N-1.$$

Then

$$||f - f^j||_{L^2(\Omega)} \leq C ||f^{(N)}||_\infty 2^{-jN},$$

$$||f - f^j||_{H^1(\Omega)} \leq C ||f^{(N)}||_\infty 2^{-j(N-1)},$$

where C is a constant depending only on N, diameter of Ω and

$$||f^{(N)}||_\infty := max_{(x,y) \in \Omega, m=0,1,\cdots,N} |\frac{\partial^N f}{\partial x^m \partial y^{N-m}}(x, y)|.$$

A new scheme of interpolation of a multivariate function with the tensor products of compactly supported multi-scaling functions was introduced in [LX]. The idea is to choose weighted sample values of a given function by suitable weights. The advantages of multiwavelets versus scalar wavelets are possessing symmetry, orthogonality, short support and high approximation order [GHM],[SN],[SS]. A similar result to Theorem 1.1 in multi-scaling case will be mentioned in section 3, namely, given a multivariate function $f \in C_0^N$, the interpolation scheme provides, at scale

2^j, an approximation order N in the L_2 space or an approximation order N-n in the Sobelev space H^n if and only if the multi-scaling function has the accuracy N.

In the context of the multi-scaling functions, one usually deals with a matrix dilation (refinement) equation:

$$\Phi(t) = \sum_k C_k \Phi(2t - k). \tag{1.3}$$

Here $\Phi(t) = [\Phi_0(t), \cdots, \Phi_{r-1}(t)]^T \in L^2$ is a multi-scaling function and C_k are r by r matrices with constant elements. If $r = 1$, C_k are real numbers and $\Phi(t) = \Phi_0(t)$ which is called scalar scaling function. A characterization of the multi-scaling function is described in the next section.

2. **Approximation accuracy of multi-scaling functions**

A multi-scaling function $\Phi(x) = [\Phi_0(x), \Phi_1(x), \cdots, \Phi_{r-1}(x)]^T$ is said to have *approximation accuracy* N if each polynomial x^p, $p = 0, \cdots, N-1$, is a locally finite series of the integer translates $\Phi(x+k)$, $k \in Z$, i.e.

$$x^p = \sum_k y_k^{(p)} \Phi(x+k) \quad a.e., \quad p = 0, \cdots, N-1 \tag{2.1}$$

where $y_k^{(p)}$ are constant row vectors and $y_0^{(p)}$ are called *starting vectors*. In [HSS], Heil, Strang and Strela show the following necessary condition of linear independency, namely, if Φ has accuracy N and $\{\Phi_\alpha\ (x+k),\ \alpha = 0, \cdots, r-1,\ k \in Z\}$ are linearly independent then

$$y_k^{(p)} = \sum_{m=0}^{p} \binom{p}{m} (-k)^{p-m} y_0^{(m)}, \quad p = 0, \cdots, N-1. \tag{2.2}$$

Hence (2.2) may slightly be weaker than the independency assumption. An equivelent statement of approximation accuracy is stated as follows[LX].

Theorem 2.1. *Suppose that a multiscaling function $\Phi(x)$ is compactly supported. Then $\Phi(x)$ has approximation accuracy N and satisfies (2.2) if and only if*

$$\sum_k \sum_{m=0}^{p} \binom{p}{m} (-1)^m\ (x+k)^{p-m}\ y_0^{(m)} \Phi(x+k) = \delta_{po} \quad a.e., \quad p = 0, 1, \cdots, N-1. \tag{2.3}$$

This theorem provides a key technique to establish a necessary and sufficient conditions for an approximation theorem by using multi-scaling function interpolation which is stated in the next section.

3. **Multi-scaling Function Interpolation (MSF Interpolation)**

The interpolation formula (1.2) uses the following type of sample values of f,

$$f_{kl}^j = \frac{1}{2^j} f\left(\frac{k+\lambda}{2^j}, \frac{l+\lambda}{2^j}\right). \tag{3.1}$$

It was shown that the asymptotic behavior of the interpolation is not as nice as the L^2-projection [D]. The convergence rate was improved by introducing the Multi-scaling Function Interpolation (MSF interpolation)[LX]. Lin and Xiao show that the necessary and sufficient condition for the asymptotic behavior of MSF interpolation with error 2^{-jN} is that Φ has approximation accuracy N. The idea is to choose the sample values by using suitable weighted average of the values of given function at certain sampling points.

Let $\lambda_{m\alpha}, u_{m\alpha},\ \alpha = 0, 1, \cdots, r-1, m = 0, 1, \cdots, M-1$, be constants. We choose the sampling values as follows.
For $j \in N$, let

$$f^j_{k,\alpha,l,\beta} = \sum_{n=0}^{M-1} \sum_{m=0}^{M-1} u_{n\alpha}\ u_{m\beta}\ f(\frac{k+\lambda_{n\alpha}}{2^j}, \frac{l+\lambda_{m\beta}}{2^j}) \tag{3.2}$$

and

$$f^j(x,y) = \frac{1}{2^j} \sum_k \sum_l \sum_{\alpha=0}^{r-1} \sum_{\beta=o}^{r-1} f^j_{k,\alpha,l,\beta}, \Phi^j_{\alpha k}(x)\Phi^j_{\beta l}(y). \tag{3.3}$$

The weighted coefficients $u_{m\alpha}$ are related to the starting vectors as follows:

$$y_0^{(p)} = (\sum_{m=0}^{M-1} u_{m0}\lambda^p_{m0}, \sum_{m=0}^{M-1} u_{m1}\lambda^p_{m1}, \cdots, \sum_{m=0}^{M-1} u_{m,r-1}\lambda^p_{m,r-1}), \tag{3.4}$$

$$p = 0,\ \cdots,\ N-1, \qquad \lambda^0_{m\alpha} = 1.$$

Theorem 3.1. *Suppose $\Phi(x)$ is compactly supported. Given $\lambda_{m\alpha},\ u_{m\alpha}$ and define f^j as (3.3). Then, for all $f \in C_0^N(R^2)$,*

$$||f - f^j||_{L^2} \le C2^{-jN}, \tag{3.5}$$

if and only if Φ has approximation accuracy N and its associate vectors $y_k^{(p)}$ satisfy (2.2), (3.4). The constant C is independent of j.

Theorem 3.2. *Suppose a multi-scaling function $\Phi(x)$ is compactly supported and $\Phi(x) \in H^n(R),\ n \ge 0$. If $\Phi(x)$ has approximation accuracy N $(N \ge n)$ and satisfies (2.2) and (3.4) then, for all $f \in C_o^N(R^2)$,*

$$||f - f^j||_{H^n} \le C2^{-j(N-n)} \tag{3.6}$$

where C is independent of j. Conversely, if (3.6) holds, then Φ has approximation accuracy N.

The constants $\lambda_{m\alpha}, u_{m\alpha}$ in Theorem 3.1 give us some flexibility in choosing the sampling points and the weighted coefficients. Some special cases deserve our attetion [LX], which are stated below. We first consider $M = N$, and choose λ's so that, for each $\alpha, \lambda_{m\alpha} \ne \lambda_{n\alpha}, m \ne n$, then one can always solve (3.4) for $u_{m\alpha}$. For example, we set

$$\lambda_{m\alpha} = m, \lambda_{m\alpha} = -[\frac{M}{2}] + m, \quad \lambda_{m\alpha} = M - m \quad or \quad \lambda_{m\alpha} = m - M.$$

In these settings, the sampling points can be kept at lever j. In particular, the following is true when $\lambda_{m\alpha} = m$.

Corollary 3.1. *Suppose a multi-scaling function $\Phi(x)$ is compactly supported and has approximation accuracy N with starting vectors $y_0^{(0)}, \cdots, y_0^{(N-1)}$.*
Let vectors $Z_0, \cdots, Z_{N-1}$ satisfy

$$\sum_{n=0}^{N-1} Z_n n^p = y_0^{(p)}, \quad p = 0, \cdots, N-1.$$

Then for all $f \in C_0^N(R^2)$,

$$||f(x,y) - \frac{1}{2^j} \sum_k \sum_l \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f(\frac{k}{2^j}, \frac{l}{2^j}) Z_n \Phi_{k-n}^j Z_m \Phi_{l-m}^j \ (y)||_{L^2} \leq C 2^{-jN}$$

and

$$||f(x,y) - \frac{1}{2^j} \sum_k \sum_l \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f(\frac{k}{2^j}, \frac{l}{2^j}) Z_n \Phi_{k-n}^j Z_m \Phi_{l-m}^j \ (y)||_{H^n} \leq C 2^{-j(N-n)}$$

provided $\Phi(x) \in H^n$ and $N \geq n$ where C is independent of j.

We next consider M=1, N= 2, then (3.4) is reduced to

$$y_0^{(0)} = (u_{00}, \cdots, u_{0,r-1})$$

and

$$y_0^{(1)} = (\lambda_{00} u_{00}, \cdots, \lambda_{0,r-1} u_{0,r-1}).$$

Corollary 3.2. *Let $\Phi(x)$ be compactly supported and have approximation accuracy 2 with starting vectors*

$$y_0^{(0)} = (u_0, \cdots, u_{r-1})$$

and

$$y_0^{(1)} = (v_0, \cdots, v_{r-1})$$

If $u_\alpha \neq 0, \quad \alpha = 0, \cdots, r-1$ then for all $f \in C_0^N(R^2)$

$$||f(x,y) - \frac{1}{2^j} \sum_k \sum_l \sum_{\alpha=0}^{r-1} \sum_{\beta=0}^{r-1} u_\alpha u_\beta f(\frac{k + \frac{v_\alpha}{u_\alpha}}{2^j}, \frac{l + \frac{v_\beta}{u_\beta}}{2^j}) \Phi_{\alpha k}^j(x) \Phi_{\beta l}^j(y)||_{L^2} \leq C 2^{-2j}, \tag{3.7}$$

Moreover, if in addition $\Phi \in H^1$, then

$$||f(x,y) - \frac{1}{2^j} \sum_k \sum_l \sum_{\alpha=0}^{r-1} \sum_{\beta=0}^{r-1} u_\alpha u_\beta f(\frac{k + \frac{v_\alpha}{u_\alpha}}{2^j}, \frac{l + \frac{v_\beta}{u_\beta}}{2^j}) \Phi_{\alpha k}^j(x) \Phi_{\beta l}^j(y)||_{H^1} \leq C 2^{-j},$$

where C is independent of j.

4. Applications

In practice, one can use the following scaling functions for the multiscaling function interpolation as stated in Section 3.

4.1 Two Dimensional Case

We use corollary 3.1 and multi-scaling function $\phi(t) = (\phi_0(t), \phi_1(t))$,

$$\phi_0(t) = \begin{cases} -2t^3 + 3t^2 & t \in [0,1), \\ (t-2)^2(2t-1) & t \in [1,2], \\ 0 & \text{otherwise.} \end{cases}$$

$$\phi_1(t) = \begin{cases} -3t^3 + 3t^2 & t \in [0,1), \\ (t-2)^2(-3t+3) & t \in [1,2], \\ 0 & \text{otherwise.} \end{cases}$$

ϕ has accuracy 4 [PS], and its starting vectors are

$$y_0^{(0)} = (1,0), \quad y_0^{(1)} = (1,-\frac{1}{3}), \quad y_0^{(2)} = (1,-\frac{2}{3}), \quad y_0^{(3)} = (1,-1).$$

Follow Corollary 3.1, we have

$$Z_0 = (0,\frac{1}{9}), \quad Z_1 = (1,\frac{1}{6}), \quad Z_2 = (0,-\frac{1}{3}), \quad Z_3 = (0,\frac{1}{18}).$$

Let

$$\triangle f_j = ||f(x,y) - \frac{1}{2^j}\sum_k \sum_l \sum_{n=0}^{3}\sum_{m=0}^{3} f(\frac{k}{2^j},\frac{l}{2^j}) Z_n \phi_{k-n}^j(x) Z_m \phi_{l-m}^j(y)||_{L^2([0,1]\times[0,1])}$$

4.2 One Dimensional Case

The one dimensional version of (3.7) is

$$||f(x) - \frac{1}{2^j}\sum_k \sum_{\alpha=0}^{r-1} u_\alpha f(\frac{k+\frac{v_\alpha}{u_\alpha}}{2^j})\Phi_{\alpha k}^j(x)||_{L^2} \le C 2^{-2j}.$$

Use this interpolation formula, we approximate $F(x) \in C_0^2(R^1)$ as follows.

Consider the following multi-scaling function $\Phi(t) = (\Phi_0(t), \Phi_1(t))$.

$$\Phi_0(t) = \begin{cases} -5t^3 + 6t^2 & t \in [0,1), \\ (2-t)^2(-t+2) & t \in [1,2], \\ 0 & \text{otherwise.} \end{cases}$$

$$\Phi_1(t) = \begin{cases} -t^3 & t \in [0,1), \\ (2-t)^2(-5t+4) & t \in [1,2], \\ 0 & \text{otherwise.} \end{cases}$$

The first two new starting vectors of $\Phi(t)$ are $(\frac{1}{2},-\frac{1}{2})$ and $(\frac{1}{3},-\frac{2}{3})$. Let

$$\triangle F_j = ||F - F^j||_{L^2[0,1]}.$$

where

$$F^j(x) = \sum_{k=-2}^{2^j} \{\frac{1}{2}F(\frac{k+\frac{2}{3}}{2^j})\Phi_0(2^j x - k) - \frac{1}{2}F(\frac{k+\frac{4}{3}}{2^j})\Phi_1(2^j x - k)\}.$$

5. **Multiscaling Approximation**

Let the kernel of (1.1) satisfy the following conditions

(5.1) $\sup_{0\le s\le 1}\int_0^1 |k(s,t)|dt < \infty$

(5.2) $\lim_{\delta\to 0}\int_0^1 |k(s+\delta,t)-k(s,t)|dt = 0 \quad 0\le s\le 1$

(5.3) $k(s,t)$ is symmetric, i.e. $k(s,t)=k(t,s)$

Consider the multiresolution analysis, $\pi_j K u_j = \lambda_j u_j, \quad u_j \in V_j$ where π_j is a projection operator onto V_j.

From condition (5.1) above, $\lambda \neq 0$, we have

$$\lambda_j(u, {u^i}_j) = (u, \lambda_j {u^i}_j) = (u, K{u^i}_j).$$

Thus

$$\lambda_j = \frac{(u, K{u^i}_j)}{(u, {u^i}_j)} = (u, K\tilde{u}_j)$$

where $\tilde{u}_j = \frac{{u^i}_j}{(u,{u^i}_j)}, {u^i}_j$ is the mutiscale interpolation of u.

On the other hand,

$$\lambda(u, u_j) = (\lambda u, u_j) = (Ku, u_j)$$
$$\lambda = \frac{(Ku, u_j)}{(u, u_j)} = (Ku, \hat{u}_j)$$

where $\hat{u}_j = \frac{u_j}{(u,u_j)}$.

Hence

$$\begin{aligned}\lambda - \lambda_j &= (Ku, \hat{u}_j) - (u, K\tilde{u}_j)\\ &= (u, K\hat{u}_j) - (u, K\tilde{u}_j)\\ &= (u, K(\hat{u}_j - \tilde{u}_j))\end{aligned}$$

where we use the symmetric property of the kernel (condition 5.3).

Also,

$$||\hat{u}_j - \tilde{u}_j|| \le ||u - \hat{u}_j|| + ||u - \tilde{u}_j||.$$

We therefore have the following estimate.

Theorem 5.1.

Following the above assumptions, we have

$$|\lambda - \lambda_j| \le C2^{-j}$$

for some constant C.

The above theorem is basically based on the MRA approximation, not really based on the scaling function interpolation. However, it is conceivable the multiscaling method is more efficient and one can apply the same technique for solving higher dimensional problems as well. It would be interesting to produce some examples by applying the scaling function interpolation.

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On Numerical Aspects of a Biorthogonal Wavelet Basis

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Abstract

Wavelet methods have been of considerable interest in many engineering applications such as signal and image processing. However, like other classic orthogonal expansions, wavelet expansions also exhibit Gibbs phenomenon around discontinuities of the original signal. In a previous work, summability methods for removing the Gibbs phenomenon in the orthogonal wavelet expansion functions with jump discontinuous were introduced. These summability methods are based on replacing a conventional wavelet basis by a biorthogonal basis. This article is devoted to the discussion of the computational aspects in constructing such biorthogonal wavelet bases. An approach of linking conventional wavelet bases and their induced biorthogonal bases via a group of conversion coefficients is introduced. Numerical examples are provided to demonstrate the method.

1 Introduction

Wavelet analysis has numbers of notable advantages and has shown promising results in many engineering applications such as signal and image processing. However, like almost any other classic orthogonal (or biorthogonal) expansions, Gibbs phenomenon ([3], [4], [10] and [5]) also occurs in both discrete (series) and continuous (integral transform) wavelet expansions ([9], [13], [11] and [8]). This is true for time-limited wavelets, such as Daubechies wavelets, and for bandlimited wavelets, such as Meyer wavelets. In classical orthogonal systems, various summability methods are used to get rid of unwanted Gibbs phenomenon [7], [22]. A positive kernel is usually constructed

to replace the oscillatory reproducing kernel. An expansion associated with a positive kernel therefore does not show Gibbs phenomenon [22]. Similarly, to remove the Gibbs phenomenon in the orthogonal wavelet expansion some summability methods have been introduced in the previous work [15], [16], [17]. These summability methods are based on replacing a conventional wavelet basis by its induced biorthogonal wavelet basis (called by positive wavelet basis). The scaling subspace of the induced wavelet basis is associated with a positive reproducing kernel.

Recently, these methods have attracted some attention in engineering community [20]. In the practical world, it is inevitable that a signal exhibits singularities induced by incompatible initial data or flurried by noise. Signals are modeled mathematically by functions with jump discontinuities [1]. Digital filtering via wavelet transform can be plagued by such phenomenon. Similarly, it also appears in data compression using wavelet threshholding. To remove or eliminate Gibbs oscillation has to be considered when quality (or resolution) is crucial in certain applications.

In order to make these summability methods useful in engineering practice, an effective algorithm for computing the induced positive basis is essential. This article is devoted to such purpose. It discusses of the numerical aspects related to the construction of the induced basis. We restrict our attention to the case of Abel like summability method for wavelets with compact support. After the mother wavelet and its dual associated with the positive scaling function been introduced, we discuss how these bases are used in analysis and synthesis algorithms for functions in $L_2(R)$. An approach to link a conventional wavelet basis and its induced positive basis via a group of connection coefficients is developed. In numerical examples, we shall be interested mainly in the most popular compactly supported wavelets - Daubechies wavelets. By the end, we briefly remark on the method and our further study.

2 Background

We briefly introduce some notations and definitions which are required in the subsequent discussions. The curious readers can easily find references for orthogonal wavelet theory in [2] (Chapter 5) or [18] (Chapter 3), for example.

Definition 1 *Multiresolution Analysis (MRA). A multiresolution analysis of* $L^2(\mathbb{R})$ *is an increasing sequence* V_j, $j \in \mathbb{Z}$ *of closed linear subspaces of*

$L^2(\mathbb{R})$ *with the following properties*

$$\begin{array}{ll}(i) & \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R}) \\ (ii) & \bigcap_m V_m = \{0\}, \quad \overline{\bigcup_m V_m} = L^2(\mathbb{R}) \\ (iii) & f \in V_m \Leftrightarrow f(2\cdot) \in V_{m+1}, \text{ for all } f \in L^2(\mathbb{R}), \text{ and all } m \in \mathbb{Z} \\ (iv) & f(x) \in V_0 \Leftrightarrow f(x-k) \in V_0, \text{ for all } f \in L^2(\mathbb{R}), \text{ and all } m \in \mathbb{Z}\end{array} \tag{1}$$

For a given MRA, there exists (not necessarily unique) a function $\phi \in V_0$, such that

$$\{\phi(t-n), n \in \mathbb{Z}\}$$

forms an orthonormal basis of V_0. The function ϕ is called a scaling function of the given MRA. The functions

$$\{\phi_{m,n}(t) = 2^{m/2}\phi(2^m t - n),\ m,\ n \in \mathbb{Z}\}$$

form an orthonormal basis of V_m.

The mother wavelet is defined by the dilation equations:

$$\begin{array}{ll}(i) & \phi(t) = \sum_k \sqrt{2} h_k \phi(2t-k), \\ (ii) & \psi(t) = \sum_k \sqrt{2}(-1)^k h_{1-k}\phi(2t-k),\end{array} \quad h_k \in l^2(R) \tag{2}$$

with Fourier transform:

$$\hat{\psi}(w) = e^{i\pi w/2}\overline{m_0(w/2+\pi)}\hat{\phi}\left(\frac{w}{2}\right)$$

where

$$m_0(w) = \sum_k \frac{h_k}{\sqrt{2}} e^{-iwk/2}$$

It is remarkable that each of the spaces V_m in a multiresolution analysis is a reproducing kernel Hilbert space with reproducing kernel (RK) defined by:

$$q_0(t,s) = \sum_{n=-\infty}^{\infty} \phi_{0,n}(t)\overline{\phi_{0,n}(s)},$$

and in general at level m

$$q_m(t,s) = 2^m q(2^m t, 2^m s) = \sum_{n=-\infty}^{\infty} \phi_{m,n}(t)\overline{\phi_{m,n}(s)}\ .$$

The kernel is said to be positive if for any positive function f,

$$f_0(t) = \int_{-\infty}^{\infty} q(s,t)f(s)ds > 0.$$

By using the reproducing kernel of V_m, we can write the projection of a function $f \in L^2(R)$ onto V_m as:

$$\begin{aligned} f_m(t) &= P_m f = \sum_{n=-\infty}^{\infty} (f, \sum_{j=-\infty}^{\infty} \phi_{m,j})\phi_{m,n}(t) \\ &= \int_{-\infty}^{\infty} q_m(t,s)f(s)ds. \end{aligned} \tag{3}$$

There are two other related discrete expansions. One is the projection onto the wavelet subspace W_m :

$$\begin{aligned} f^m(t) &= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (f, \psi_{k,n})\psi_{k,n}(t) \\ &= \int_{-\infty}^{\infty} r_m(t,s)f(s)ds \end{aligned} \tag{4}$$

where $r_m(t,s)$ is the reproducing kernel of the subspace W_m. Using (3), we can rewrite (4) to get a hybrid expansion

$$f^m(t) = \sum_{n=-\infty}^{\infty} < f, \sum_{n=-\infty}^{\infty} \phi_{0,n} > \phi_{0,n}(t) + \sum_{k=-\infty}^{m} \sum_{n=-\infty}^{\infty} (f, \psi_{k,n})\psi_{k,n}(t)$$

As shown by [6], the standard RK is never positive for any continuous scaling functions. The RK associated with Harr wavelet is positive while the Harr scaling function has jump discontinuity itself. Consequently, all wavelet expansions present Gibbs phenomenon inevitably. This has been proved in [9], [13]. There exist some substantial differences between the Fourier basis and a wavelet basis. As well known, wavelet bases do not possess shifting invariant properties like the Fourier basis. Therefore, the behavior of the wavelet expansion at jump discontinuities depends on the location of the discontinuities. In wavelet case, to describe the phenomenon qualitatively, Gibbs function (defined below) other than "Gibbs constant" (in Fourier case) is used.

Definition 2 *(Gibbs function) The Gibbs function $G(t,a)$, is defined as:*

$$G(t,a) = \int_a^\infty q(t+a,s)ds, \tag{5}$$

where a is the location parameter.

Unlike the situation in Fourier expansion, the convergence can not be guaranteed in a wavelet expansion at the jump discontinuity of the function. As a result, Gibbs effects for a wavelet system is more difficult to manipulate quantitatively.. There are some discussions about the asymptotic behavior of the Gibbs function at $a = 0$ in the literature [13],[21]. Nevertheless the knowledge about the size of over shoot or under shoot is much limited. Figure 1 shows Gibbs function associated with Coiflets, $N = 2$, and $N = 3$. Figure 2 shows the Gibbs function for Coiflets, $N = 4$ comparing with Gibbs function for Daubechies wavelets, $N = 4$. The parameter N is related to the support interval of the scaling function. For Daubechies wavelet, the support interval is $[0, 2N-1]$, for Coiflets, the support interval is $[0, 3N-1]$.

To remove Gibbs phenomenon in the orthogonal wavelet expansion for functions with jump discontinuity, summability methods have been introduced in the previous work [15], [16], [17]. The follow theorem has been proved,

Theorem 1 *(Walter and Shen [15]) Let $f \in L^1(\mathrm{R})$; let*

$$k_r(t,s) = \sum_{n=-\infty}^{\infty} \rho_r(t-n)\rho_r(s-n), \tag{6}$$

where

$$\rho_r(t) = \frac{1-r}{1+r} \sum_{n=-\infty}^{\infty} r^{|n|}\phi(t-n) \tag{7}$$

$$f_m^r(t) = \int_{-\infty}^{\infty} k_{r,m}(t,s)f(s)ds; \tag{8}$$

then $f_m^r \in V_m$ and

(i) if $M_1 \le f(t) \le M_2$ for $t \in \mathrm{R}$, then $M_1 \le f_m^r(t) \le M_2$ for $t \in \mathrm{R}, m \in \mathrm{Z}$,

(ii) if $M_3 \le f(t) \le M_4$ for $t \in [a,b]$, then for each $\epsilon > 0, \delta > 0$, there is an m_0, such that for $t \in (a+\delta, b-\delta), M_3 - \epsilon \le f_m^r(t) \le M_4 + \epsilon$, for $m \ge m_0$.

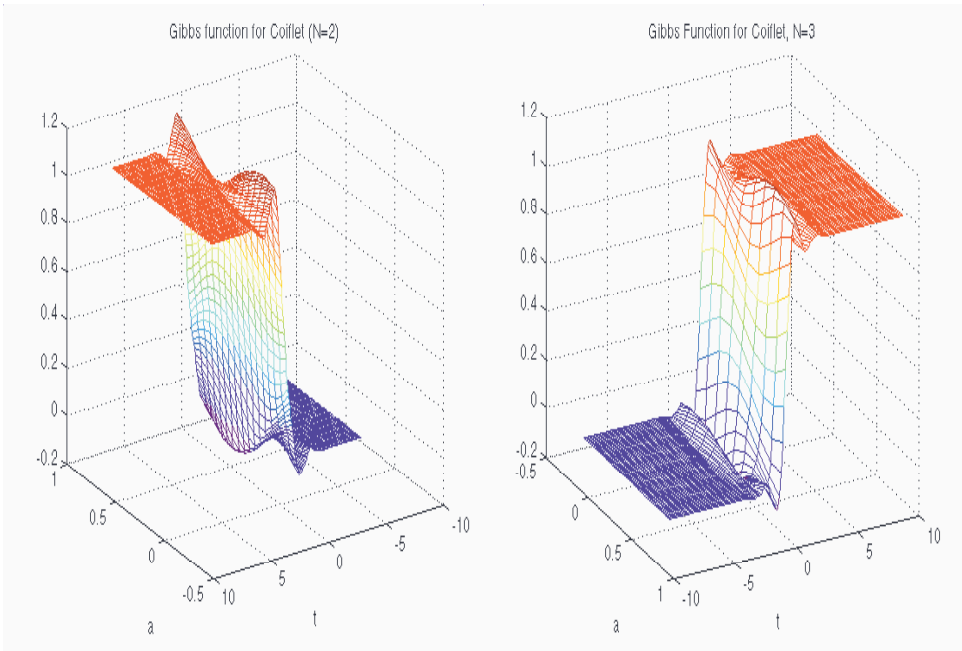


Figure 1: A comparison of Gibbs functions for Coiflets with different support in lengths. Left: $N = 2$, Right: $N = 3$.

Property (ii) in Theorem 1 guarantees that expansions using kernel k_r do not show Gibbs phenomenon. This result is useful in some applications of different areas. In theory, formula (8) can be used to compute the expansion coefficients. However, in practice, the integral format is not convenient. On the other hand, in order to benefit to the multiscale nature of wavelet bases, we have to construct "wavelets" associated with the induced positive scaling function ρ_r. In what follows, we will introduce the associated mother wavelet ξ_r and its dual function $\widetilde{\xi}_r$. The two pairs of biorthogonal bases, $\{\rho_r, \widetilde{\rho}_r\}$ and $\{\xi_r, \widetilde{\xi}_r\}$, consist of a MRA of L_2. We refer this set of biorthogonal wavelet basis as induced positive wavelet basis. The parameter r is called by regularization parameter. The word "positive" is sort of confusing. It refers to the RK associated with the scaling function (not the mother wavelet) is positive.

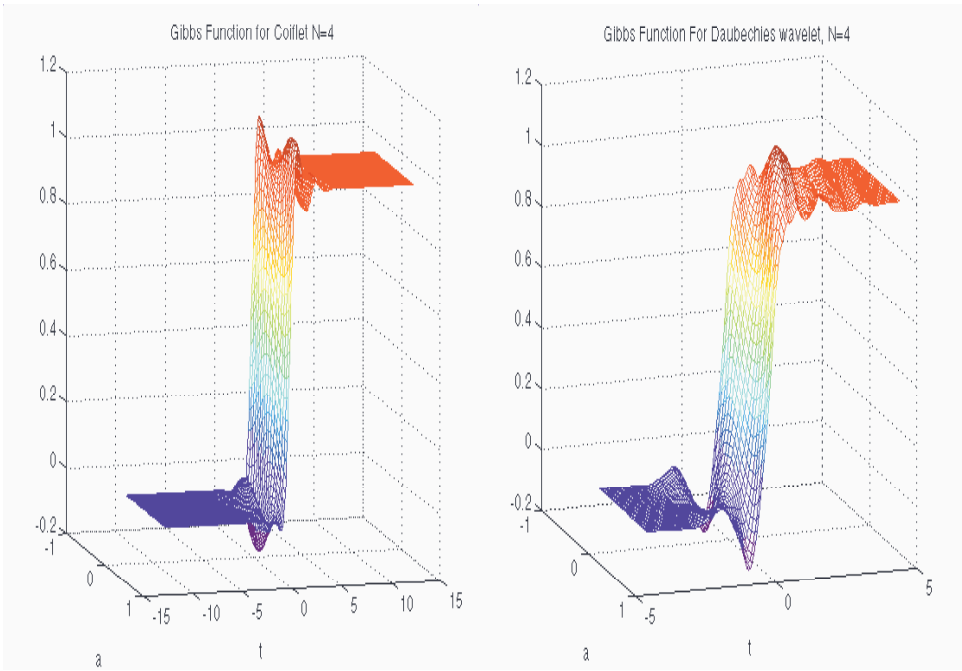


Figure 2: A comparison of Gibbs functions of Coiflet and Daubechies wavelet. Left: Coiflet, $N = 4$. Right: Daubechies, $N = 4$.

3 Construction of the induced positive wavelet system

We begin with the induced positive scaling function.

3.1 The induced positive scaling function

Unfortunately, the positive scaling function ρ_r (7) is not orthogonal although the original scaling function is. The dual basis of the positive scaling function is defined in the following proposition:

Proposition 2 *The dual function of the positive scaling function ρ_r is given by,*

$$\widetilde{\rho}_r(t) = \frac{1}{2\pi(1-r^2)}[(1+r^2)\phi(t) - r\{\phi(t+1) + \phi(t-1)\}].$$

In addition if we denote

$$\widetilde{\rho}_r(t) = \sum_{k=-\infty}^{\infty} d_k^r \rho_r(t-k),$$

then

$$d_k^r = r^{|k|}(|k| + \frac{1+r^2}{1-r^2}),\ k = -\infty, ..., \infty. \tag{9}$$

Conversely, we have

$$\rho_r(t) = \sum_{k=-2}^{2} c_k^r \widetilde{\rho}_r(t-k),$$

where

$$c_0^r = \tfrac{1-4r^2+r^4}{[2\pi(1-r^2)]^2} \quad c_1^r = -\tfrac{2r(1+r^2)}{[2\pi(1-r^2)]^2} \quad c_2^r = \tfrac{r^2}{[2\pi(1-r^2)]^2} \quad c_i^r = c_{-i}^r, i = 1, 2. \tag{10}$$

Formulas (9) and (10) are the conversion coefficients between the positive scaling function and its dual.

Unlike the positive scaling function ρ_r, the dual basis $\widetilde{\rho}_r$ is compactly supported. Fig. 3 shows a pair of induced positive scaling function and its dual associated with Daubechies wavelet D_4. The regularizing parameter r should be chosen as small as possible while guarantees the positivity of the positive scaling function ρ_r. The selection of parameter r is based on experiments. By Theorem 1, it is clear that series (7) converges faster when parameter r is smaller. From numerical experiments, we also have observed that a smaller parameter r is corresponding with a more localized positive scaling function. However, when r is progressively smaller, the induced scaling function is not positive any more. Figure 4 shows the comparison among different choices of r. The bottom panel shows 3 failed cases, in which negative values are found for the scaling functions.

3.2 The induced mother wavelet

To construct mother wavelet associated with the positive scaling function, we use the fact that the induced mother wavelet is in the subspace generated by the rescaled dual positive scaling function and its shiftings, namely $\{\widetilde{\rho}_r(2t-k)\}$. We summarize the construction of the biorthogonal positive wavelet in the following proposition. From now on, we will assume the original scaling function is the scaling function of Daubechies wavelet with filter coefficients $\{h_k\}_{k=0}^{2N-1}$.

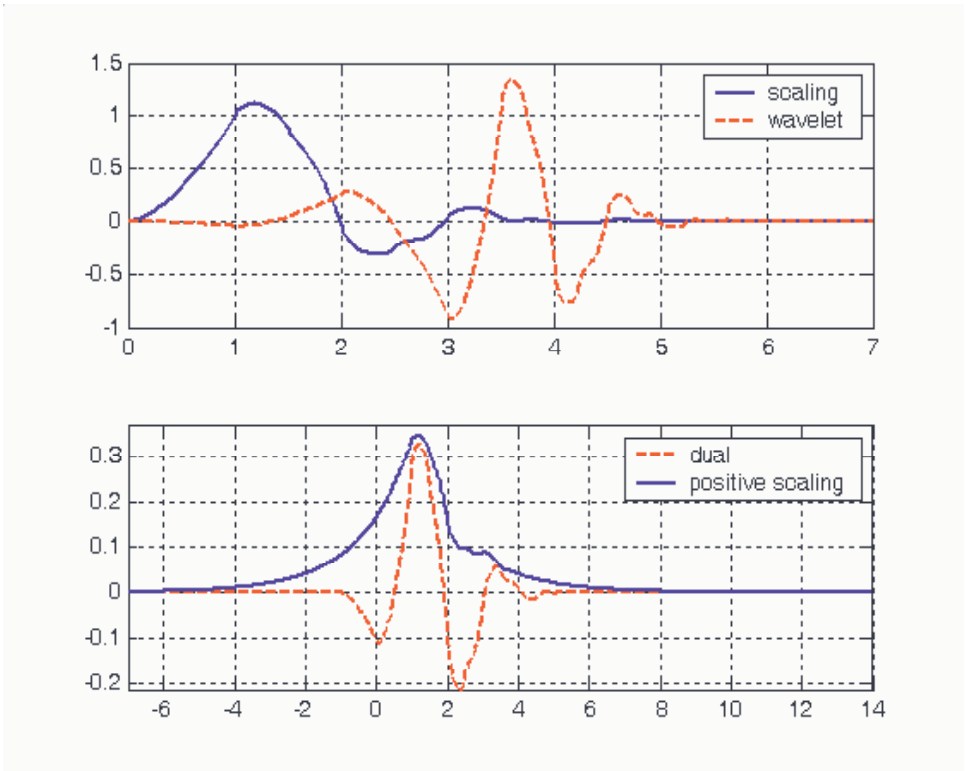


Figure 3: The positive scaling function and its dual function associated with Daubechies wavelet D_4. Top: Scaling function and mother wavelet for Daubechies wavelet D_4. Bottom: Induced positive scaling function and its dual function. $r = 0.5, N = 7, M = 13$.

Proposition 3 *The positive wavelet associated with the induced scaling function (7) is given by the following dilation equation:*

$$\xi_r(t) = \sqrt{2} \sum_{k=1}^{N+1} (-1)^k \widetilde{a}_{1-k} \widetilde{\rho}_r(2t-k). \tag{11}$$

where

$$\widetilde{a}_k = \frac{1}{2\pi(1-r^2)} \sum_{n=0}^{2N-1} (1 + r^2 - r^{|k-2-n|+1} - r^{|k+2-n|+1}) h_k.$$

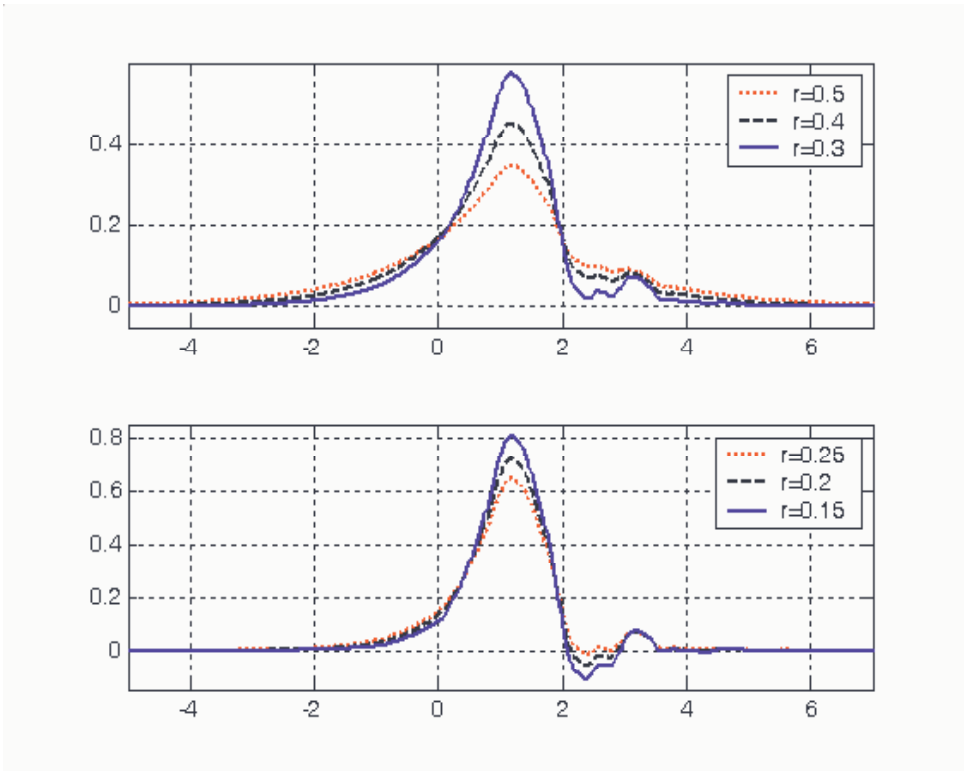


Figure 4: The choices of regularization parameter r. Top: 3 successful choices for parameter r, 0.5 (solid line), 0.4 (dashed line) and 0.3 (dotted line). Bottom: 3 unsuccessful choices for parameter r, 0.25 (solid line), 0.2 (dashed line) and 0.15 (dotted line).

Moreover, the dual function associated with the induced wavelet is given by

$$\widetilde{\xi}_r(t) = \sqrt{2} \sum_{k=-\infty}^{\infty} (-1)^k a_{1-k} \rho_r(2t-k), \tag{12}$$

where

$$a_k = \sum_{n=-\infty}^{\infty} \left(h_{k-2n} r^{|n|} \frac{1+r^2}{1-r^2} - h_{k-1-2n} \frac{r^{|n|+1}}{1-r^2} - h_{k+1-2n} \frac{r^{|n|-1}}{1-r^2} \right).$$

The following proposition relates the original mother wavelet with the induced mother wavelet.

Proposition 4 *The wavelet function and its induced positive wavelet are related by:*

$$\begin{aligned} \psi(t) &= \sum_{k=-\infty}^{\infty} \psi_k^r \xi_r(t-k) \\ \xi^r(t) &= \sum_{k=-\infty}^{\infty} \xi_k^r \psi(t-k) \end{aligned} \tag{13}$$

where

$$\begin{cases} \psi_k^r = \sum_{k=-\infty}^{\infty} [(1+r^2)h_k + rh_{k+1} + rh_{k-1})] a_{k+2n}^r; \\ \xi_k^r = \sum_{k=-\infty}^{\infty} (-1)^k \widetilde{a}_{1-k}^r \sum_{l=\mathbf{0}}^{2N-1} (-1)^{m-1} h_m r^{|2n+1-m-k|}. \end{cases}$$

The above propositions can be derived by straightforward (but rather long) calculations, we take liberty to omit their proofs accordingly.

Proposition 5 *If the dilation equation for the positive scaling function is written as,*

$$\rho_r(t) = \sqrt{2} \sum_{k=-\infty}^{\infty} a_k^r \rho_r(2t-k), \tag{14}$$

then the filter coefficients are given by,

$$a_k^r = \sum_{n=-\infty}^{\infty} \left(h_{k-2n} r^{|n|} \frac{1+r^2}{1-r^2} - h_{k-1-2n} \frac{r^{|n|+1}}{1-r^2} - h_{k+1-2n} \frac{r^{|n|-1}}{1-r^2} \right). \tag{15}$$

In addition, $\{a_k^r\}_{k=-\infty}^{\infty}$*, are exponentially decreasing as* $k \to \pm\infty$*.*

PROOF

Using the dilation equation for Daubechies scaling function, we have:

$$\phi(t-n) = \sqrt{2} \sum_{k=\mathbf{0}}^{2N-1} h_k \phi(2(t-n)-k). \tag{16}$$

Substitute (16) in (7). (15) is then derive by comparing the coefficients on both sides of (14).

Now to show the last statement in the proposition, we write,

$$\begin{aligned} a_k^r &= \sum_{n=-\infty}^{\infty} \left(h_{k-2n} r^{|n|} \frac{1+r^2}{1-r^2} - h_{k-1-2n} \frac{r^{|n|+1}}{1-r^2} - h_{k+1-2n} \frac{r^{|n|-1}}{1-r^2} \right) \\ &= a_k^{(1)} + a_k^{(2)} + a_k^{(3)}. \end{aligned} \tag{17}$$

In the case of Daubechies scaling functions, $h_n = 0$, for $n < 0$ or $n > 2N$. For $k > 2N - 1$, we have ,

$$|a_k^{(1)}| \leq \frac{1+r^2}{1-r^2} \cdot \frac{1-r^{-N}}{1-r^{-1}} \cdot \frac{1}{\sqrt{r}} r^{k/2}, k \geq 2N.$$

Similarly, we have the estimation:

$$|a_k^{(2)}| \leq \frac{r}{1-r^2} \cdot \frac{1-r^{-N}}{1-r^{-1}} \cdot \frac{1}{\sqrt{r}} r^{k/2},\ k \geq 2N,$$

$$|a_k^{(3)}| \leq \frac{r^{-2}}{1-r^2} \cdot \frac{1-r^{-N}}{1-r^{-1}} \cdot \frac{1}{\sqrt{r}} r^{k/2},\ k \geq 2N.$$

If we denote $C(r,N) = \frac{1+r^{-2}+r+r^2}{(1-r^2)\sqrt{r}} \cdot \frac{1-r^{-N}}{1-r^{-1}}$, then

$$|a_k^r| \leq C(r,N) \cdot r^{k/2}, \qquad k \geq 2N.$$

This estimate is also true if $k < 0$. For a typical case, say, if we choose $r = \frac{1}{2}$, then $C(\frac{1}{2}, 4) = 115\sqrt{2}$. In fact, in the case of Daubechies scaling functions, $h_n = 0$, for $n < 0$ or $n > 2N$. For $k > 2N - 1$.

The coefficients corresponding to different N and r (denoted as $a_k[N,r]$) are list in the Table1. Figure 4 illustrates these data graphically.

Table1. Filter coefficients a_k^r

k	$a_k[4,0.5]$	$a_k[5,0.5]$	$a_k[6,0.48]$	$a_k[8,0.42]$
15	2.48170E-2	2.63767E-2	2.07498E-2	9.12566E-3
16	1.74543E-2	1.86925E-2	1.43549E-2	5.90334E-3
17	1.24085E-2	1.31883E-2	9.95990E-3	3.83278E-3
18	8.72715E-3	9.34625E-3	6.89034E-3	2.47940E-3
19	6.20424E-3	6.59417E-3	4.78075E-3	1.60977E-3
20	4.36358E-3	4.67313E-3	3.30736E-3	1.04135E-3
21	3.10212E-3	3.29709E-3	2.29476E-3	6.76102E-4
22	2.18179E-3	2.33656E-3	1.58753E-3	4.37367E-4
23	1.55106E-3	1.64854E-3	1.10149E-3	2.83963E-4
24	1.09089E-3	1.16828E-3	7.62017E-4	1.83694E-4
25	7.75530E-4	8.24271E-4	5.28713E-4	1.19264E-4
31	9.69412E-5	1.03034E-4	5.84714E-5	8.83606E-6
38	8.52261E-6	9.12720E-6	4.47356E-6	4.23487E-7
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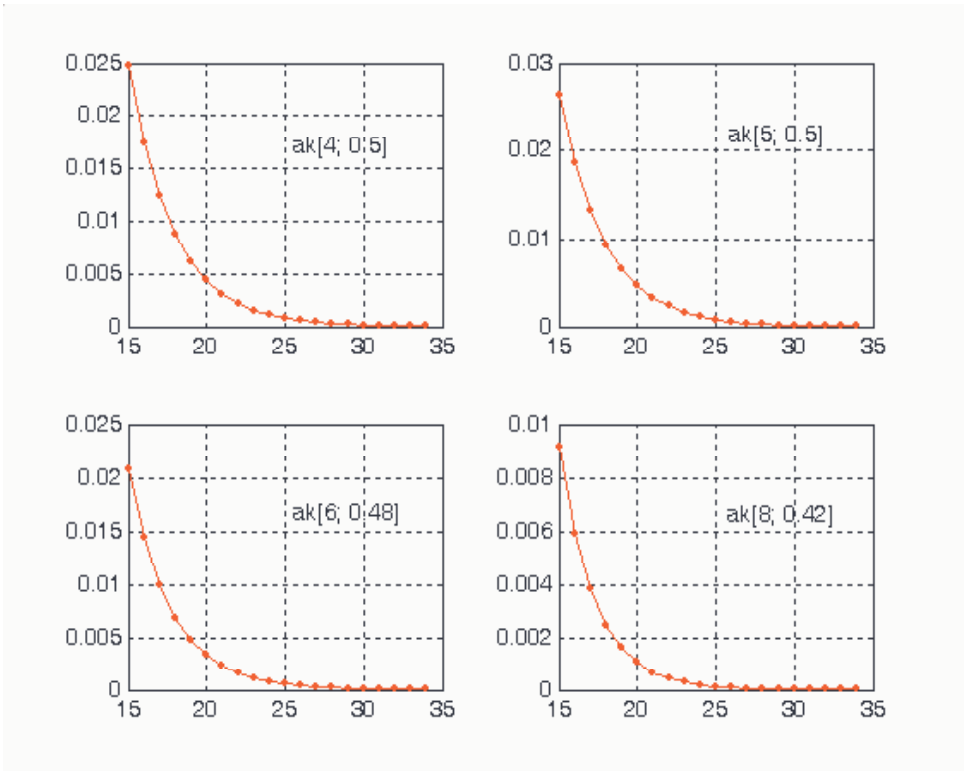


Figure 5: The decreasing rate of filter coefficients a_k for the positive scaling function induced from Daubechies wavelets. From top left, $r = 0.5, N = 4$; $r = 0.r, N = 5$; $r = 0.48, N = 6$; $r = 0.42, N = 8$.

Notice that, for every fixed k, (17) only contains a finite number of terms. By using this property, a 4-step difference scheme has been developed and studied to compute these filter coefficients numerically [16]. The scheme is iterative and requests 4 initial values to start. It is similar to any IIR (Infinite Impulse Response) filters. The positive scaling function has infinite support while its dual has finite support. Reflected from the duality, the positive mother wavelet is finite supported while its dual is not.

4 Numerical Evaluation of the induced wavelet system

To benefit from the existing software (in particularly, the popular Wavelet Toolbox in MATLAB), the central idea for computing the induced positive biorthogonal basis presented here is to make a connection between the conventional wavelet basis and its induced basis. This connection is made by computing conversion coefficients between these two sets of basis.

Before we start, we recall the following list of notations:

4.1 Notations

h_k — filter coefficients for Daubechies scaling function ϕ_4, $k = 0, 1, ..., 7$.

a_k^r — filter coefficients for summability function, which serves as the scaling function in biorthogonal expansion.

$\widetilde{a}_k^r$ — filter coefficients for dual bases.

ψ_k— conversion coefficients $\psi(t) \to \xi^r(t)$.

ξ_k^r — conversion coefficients $\xi^r(t) \to \psi(t)$.

c_k^r — dual conversion coefficients $\widetilde{\rho}_m^r(t) \to \rho_m^r(t)$.

d_k^r — dual conversion coefficients $\rho^r(t) \to \widetilde{\rho}^r(t)$.

$f_{m,k}$— coefficients in scaling expansion by using Daubechies wavelets.

$f_{m,k}^r$— coefficients in expansion by using positive kernel $k_m^r(s,t)$.

$p_{m,k}^r$— coefficients in scaling expansion by using bases $\rho_{m,n}^r(t) = 2^{\frac{m}{2}}\rho(2^m t - n)$.

$q_{m,k}^r$— coefficients in wavelet expansion by using bases $\xi_{m,n}^r(t) = 2^{\frac{m}{2}}\xi(2^m t - n)$.

4.2 Computing the conversion coefficients

The Wavelet Toolbox in Matlab is a powerful tool for the conventional wavelets transform. In this section, we describe the idea of calculating the induced positive bases via their relations to the associated conventional wavelet basis. These connections are made by a group of conversion coefficients, which can be used in converting one set of bases to the other. Consequently, we can calculate our "induced positive wavelets" by using the build-in software in Matlab.

To do so, we need compute the following two sets of coefficients:

1. 1. **Conversion coefficients between scaling functions**

a. Old to New: $\phi(t) \to \rho^r(t)$

$$a_k^r = \sum_{n=-\infty}^{\infty} \left(h_{k-2n} r^{|n|} \frac{1+r^2}{1-r^2} - h_{k-1-2n} \frac{r^{|n|+1}}{1-r^2} - h_{k+1-2n} \frac{r^{|n|-1}}{1-r^2} \right)$$

b. New to Old: $\rho^r(t) \to \phi(t)$

$$\widetilde{a}_k^r = \frac{1}{2\pi(1-r^2)} \sum_{n=0}^{2N-1} (1 + r^2 - r^{|k-2-n|+1} - r^{|k+2-n|+1}) h_k$$

1. 2. **Conversion coefficients between mother wavelets**

a. Old to New: $\psi(t) \to \xi^r(t)$

$$\psi_k^r = \sum_{k=-\infty}^{\infty} [(1+r^2)h_k + r h_{k+1} + r h_{k-1})] a_{k+2n}^r. \tag{18}$$

b. New to Old: $\xi^r(t) \to \psi(t)$

$$\xi_k^r = \sum_{k=-\infty}^{\infty} (-1) \widetilde{a}_k^r \sum_{l=0}^{2N-1} (-1)^{m-1} h_m r^{|2n+1-m-k|}.$$

4.3 Computing Expansion Coefficients

To illustrate the idea, we begin with the positive scaling expansions. We recall,

1. 1. **Conversion coefficients between the scaling expansion**

The $(m+1)th$ positive projection of f_{m+1}^r is given by (8). We start with evaluating the conversion coefficients

$$f_{m+1,k}^r = \sum_{n=-\infty}^{\infty} r^{|n-l|} f_{m+1,k} \tag{19}$$

where $f_{m+1,k}$ are the coefficients we found in last section by using Wavelet Toolbox in Matlab.

1. 2. **Conversion coefficients for decomposition**

Since $f^r_{m+1,k}(t) \in V_{m+1} = V_m \oplus W_m$, we have

$$\begin{aligned} f^r_{m+1,k}(t) &= \sum_{n=-\infty}^{\infty} p^r_{m,n}\rho^r_{m,n}(t) + \sum_{n=-\infty}^{\infty} q^r_{m,n}\xi^r_{m,n}(t) \\ &= P^r_{m,n}(t) + Q^r_{m,n}(t), \end{aligned} \tag{20}$$

where

$$p^r_{m,n} = (\frac{1-r}{1+r})^2 \sum_{k=-\infty}^{\infty} \widetilde{a}_{k-2n} f^r_{m,k} \tag{21}$$

$$q^r_{m,n} = (\frac{1-r}{1+r})^2 \sum_{k=-\infty}^{\infty} (-1)^k a_{l-k+2n} f^r_{m,k} \tag{22}$$

We can calculate $p^r_{m,n}$, $q^r_{m,n}$ by using the conventional wavelet expansion and these conversion coefficients. If we write,

$$\sum_{n=-\infty}^{\infty} q^r_{m,n}\xi^r_{m,n}(t) = \sum_{n=-\infty}^{\infty} g_{m,n}\psi_{m,n}(t), \tag{23}$$

then $g_{m,n} = \int_{-\infty}^{\infty} f(t)\psi_{m,n}(t)dt$ can be calculated easily by using Wavelet Toolbox in Matlab. The relation between $q^r_{m,n}$ and $g_{m,n}$ is given by

$$q^r_{m,n} = \sum_{l=-\infty}^{\infty} [(1+r^2)h_l + rh_{l+1} + rh_{l-1}] \sum_{k=-\infty}^{\infty} g^r_{m,n-k} a_{l+2k} \tag{24}$$

Thus, we have derived all coefficients in the following decomposition scheme,

$$\begin{array}{ccccc} & & & \nearrow & p^r_{m,n} \quad \cdots \\ f^r_{m,k} & \rightarrow & f^r_{m+1,k} & \rightarrow & q^r_{m,n} \quad \cdots \end{array}$$

Notice that this decomposition can not guarantee necessarily the heritage of the positivity.

5 Numerical Examples

We consider two numerical examples in this section.

Example 1 *In this example, we expand the characteristic function as its positive scaling function series. The characteristic function of [-1/2, 1/2] is defined as:*

$$f(t) = \begin{cases} 1, \ t \in [-1/2, 1/2] \\ 0, \ t \notin [-1/2, 1/2] \end{cases}.$$

The positive wavelet basis is induced from Daubechies wavelet $N = 4$. For the positive scaling function, we have to use its truncated version:

$$\rho_{r,L,M}(t) = \frac{1-r}{1+r} \sum_{n=-L}^{M} r^{|n|} \phi(t-n),$$

with $L = -7$, and $M = 14$. The reason for us to use the non symmetric truncation for the doubly infinite series because the Daubechies wavelet is asymmetric. Numerical results from this experiment are demonstrated in Figure 6. We observe that Gibbs effects have been removed successfully. For the comparison purpose, the scaling expansion series using Daubechies wavelet ($N = 4$) is shown on the top panel. Gibbs phenomenon can be observed clearly around points $x = -1/2$ and $1/2$.

Example 2 *In this example, we consider recover signals from their sampled data. We assume that these data is contaminated by additive Gauss noise with probability distribution $N(0, 1)$. We select the induced positive wavelet associated with Daubechies wavelet, $N = 4$. The regularization parameter is chosen as $r = 0.48$. The infinite series in (7) is truncated in the same way as in the previous example. The numerical results are shown in Fig.7. From Fig.7, we observe that the positive basis give a fairly good approximation. Gibbs effects were completely removed in all these cases. Noises are also removed from the contaminated data, consequently, the recovered signals (thin line) follow the shape of the original signals very well.*

6 Conclusion

In this paper, we have discussed some useful computational matters of a summability based biorthogonal system, namely the induced positive wavelet system. The expansions using the positive wavelets do not show Gibbs phenomenon. We then developed a practical method for computing the basis functions. The central idea of the algorithm is to make a connection between the conventional wavelet system and the induced positive wavelet system. In doing so, most of the computation can be done by using the popular Wavelet Toolbox in Matlab. Consequently, any algorithm has been developed using the conventional wavelet system can be adapted to the one using induced positive wavelet. This is convenient in the case of wavelet methods such as denoising are chosen and Gibbs phenomenon are needed to be removed. The applications of such positive biorthogonal system and the evaluation of

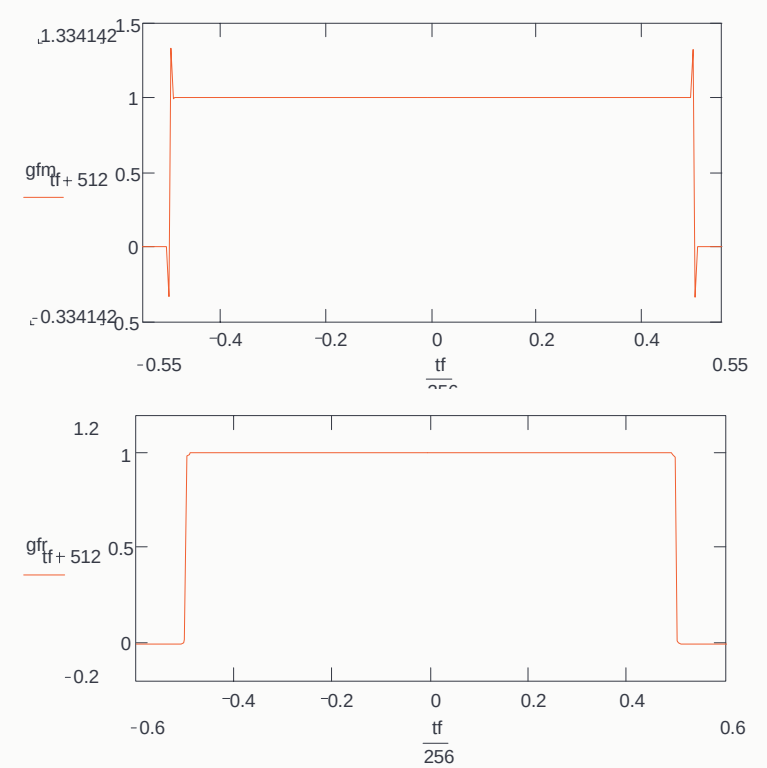


Figure 6: The approximation of square wave function by using Daubechies wavelet series ($N = 4$, $m = 8$) and its induced positive wavelet series. Top: Approximation using Daubechies wavelet. Bottom: The approximation by the induced positive wavelet series.

the effectiveness of the algorithm are under future study.

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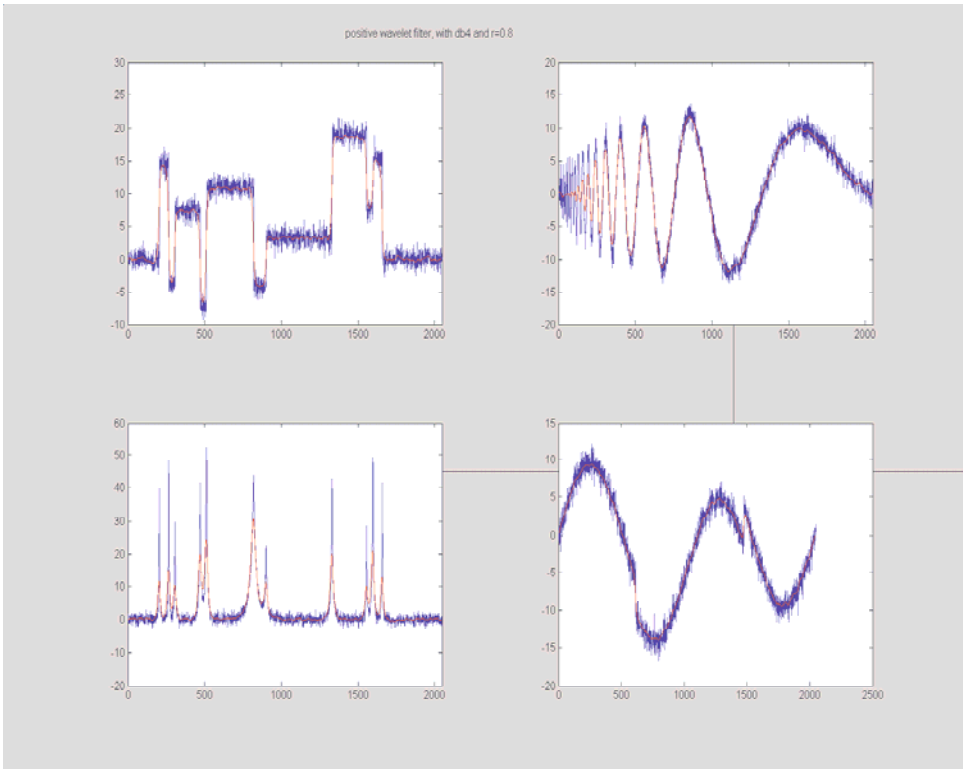


Figure 7: Signals are recovered from their sampled data contaminated by additive Gauss noise using positive wavelet series.

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2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

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